Revision on the Frequency Domain Conditions for Strict Positive Realness

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Abstract: In this paper, the necessary and sufficient conditions for strict positive realness of the rational transfer functions directly from basic definitions in the frequency domain are studied. A new frequency domain approach is used to check if a rational transfer function is a strictly positive real or not. This approach is based on the Taylor expansion and the Maximum Modulus Principle which are the fundamental tools in the complex functions analysis. Four related common statements in the strict positive realness literature which is appeared in the control theory are discussed. The drawback of these common statements is analyzed through some counter examples. Moreover a new necessary condition for strict positive realness is obtained from high frequency behavior of the Nyquist diagram of the transfer function. Finally a more simplified and completed conditions for strict positive realness of single-input single-output linear time-invariant systems are presented based on the complex functions analysis approach.

Keywords: Frequency domain analysis, maximum modulus principle, strict positive realness, taylor expansion of rational transfer function.

1. INTRODUCTION

The concept of positive realness is motivated from circuit theory. The sufficiency condition for positive realness and many of its properties are developed by Otto Brune in 1930 [1,2]. In 1963, Popov introduced the notion of hyperstability in control theory and showed that a linear time-invariant system is hyperstable system if and only if the transfer function of system is positive real, also he developed the concept of strict positive realness and showed that a linear time-invariant system is asymptotically hyperstable system if and only if the transfer function of system is strictly positive real [3]. Thus the concept "strict positive realness" of transfer functions has been extensively used in various field of control such as adaptive control [8-10], optimal control [11,12], nonlinear control [13-15], robust control [16-21] and even intelligent control [22]. Although in [31] the state space definition for strict positive realness which is called Kalman-Yakubovich-Popov (KYP) Lemma has been preferred through a lot of other definitions, the most basic definition of this concept is motivated by Popov's hyperstability theory and it is stated in frequency domain. It seems that almost all activities have been focused on the state space approaches, e.g. Kalman-Yakubovich-Popov (KYP) Lemma [23-29] and after about four decades, there is not a unique statement in the control literature which states the necessary and sufficient conditions for strict positive realness of single-input single-output system in the frequency domain. An important drawback of the state space approaches is that they involve only the proper transfer functions (this limitation can be seen in the context of Theorem 2.1 of [6] and also in the Lemma (10-1) of [7]).

In this paper, the necessary and sufficient frequency domain conditions for strict positive realness are obtained without any limitation on the relative degree of transfer function by using the frequency domain definition. The Taylor expansion approach is used to investigate four common statements in this area. A new necessary condition which imposed from high frequency part of the Nyquist diagram is deduced. Finally a more simplified and completed conditions for strict positive realness are presented based on the complex analysis approach.

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2. BASIC DEFINITIONS

Suppose $G(s)$ denotes a rational transfer function with real coefficients, then we have the following definitions.

Definition 1 [7]**:** $G(s)$ is positive real (PR) if and only if:

1) $G(s)$ is analytic in $\mathrm{Re}[s] > 0$,
2) Any pure imaginary pole of $G(s)$ is a simple pole and the associated residue is positive,
3) For all real $\omega \geq 0$ for which $j\omega$ is not a pole of $G(s)$, the inequality $\mathrm{Re}[G(j\omega)] \geq 0.$ is satisfied.

Definition 2 [3]**:** $G(s)$ is strictly positive real (SPR) if $G(s-\varepsilon)$ is positive real for sufficiently small $\varepsilon > 0$.

Combination of these definitions implies the following definition:

Definition 3: $G(s)$ is SPR if and only if for sufficiently small $\varepsilon > 0$:

1) $G(s-\varepsilon)$ is analytic in $\mathrm{Re}[s] > 0$,
2) Any pure imaginary pole of $G(s-\varepsilon)$ is a simple pole and the associated residue is positive,
3) For all real $\omega \geq 0$ for which $j\omega$ is not a pole of $G(s-\varepsilon)$, the inequality $\mathrm{Re}[G(j\omega-\varepsilon)] \geq 0$ is satisfied.

3. TAYLOR EXPANSION APPROACH

Suppose $G(s)$ is a real rational function of the complex variable $s = \sigma + j\omega$, as shown in (1).

$$G(s) = k\frac{b(s)}{a(s)} = k\frac{s^m + b_1 s^{m-1} + \ldots + b_m}{s^n + a_1 s^{n-1} + \ldots + a_n},\ k > 0. \quad (1)$$

Lemma 1: $G(s)$ is SPR if and only if:

1) $G(s)$ is analytic in $\mathrm{Re}[s] \geq 0$,
2) $n - m \geq -1$ and $k > 0$ for $n - m = -1$,
3) $\mathrm{Re}_{\varepsilon\to 0^+}[G(j\omega-\varepsilon)] \geq 0,\ \forall \omega \geq 0.$

Proof: Suppose definition (3) is as conditions for strict positive realness, then the phrase "$G(s-\varepsilon)$ is analytic in $\mathrm{Re}[s] > 0$" is equivalent to the phrase "$G(s)$ is analytic in $\mathrm{Re}[s] > -\varepsilon$". It is obvious that a real rational transfer function of complex variable $s = \sigma + j\omega$ is analytic in the whole complex plane except in its poles. Now suppose $G(s)$ be analytic in region $\mathrm{Re}[s] \geq 0$ and the nearest pole to the imaginary axis has a real part equal to –p*, we can always select ε such that it satisfies the inequality $\varepsilon <$ p*. Thus the phrase "$G(s-\varepsilon)$ is analytic in $\mathrm{Re}[s] > 0$" is equivalent to the phrase "$G(s)$ is analytic in $\mathrm{Re}[s] \geq 0$". It is clear that the second condition in definition (3) restricts the relative degree of $G(s)$ to $n - m \geq 1$, because only a simple pole at infinity is admissible and if the relative degree of $G(s)$ is equal to minus one, then the positivity of k is necessary to guarantee the positivity of the associated residue of this simple pole at infinity. Also the third condition can be restated as appear in Lemma 1.

We know that the Taylor series of a rational transfer function $G(s)$ is valid on the whole complex plane, except on the poles of $G(s)$. The first condition in the Lemma 1, guarantees the validity of Taylor expansion of $G(s)$ on the imaginary axis, thus Lemma 1 can be restated as following Lemma:

Lemma 2: $G(s)$ is SPR if and only if:

1) $G(s)$ is analytic in $\mathrm{Re}[s] \geq 0$,
2) $n - m \geq -1$ and if $n - m = -1$ then $k > 0$,
3) For each $\omega \geq 0$ inequality:

$$\lim_{\varepsilon\to 0^+} \mathrm{Re}[G(j\omega) - \varepsilon G'(j\omega) + \frac{1}{2!}\varepsilon^2 G''(j\omega) \mp \ldots] \geq 0,$$

is satisfied, where $G^{(k)}(j\omega) = \left.\dfrac{d^k}{ds^k}G(s)\right|_{s = j\omega}.$

Suppose $G_e(s)$ = (1/2) [$G(s)$ +G (-s)] and $G_o(s)$ = (1/2) [$G(s)$-G (-s)] are the even and odd parts of $G(s)$ respectively. Since the derivative of an even function is odd and the derivative of an odd function is even, it is easy to verify that:

$$\begin{aligned}&\mathrm{Re}[G(j\omega-\varepsilon)]\\ &= \mathrm{Re}[G(j\omega) - \varepsilon G'(j\omega) + \frac{1}{2!}\varepsilon^2 G''(j\omega) \mp \cdots] \quad (2)\\ &= G_e(j\omega) - \varepsilon G_o'(j\omega) + \frac{1}{2!}\varepsilon^2 G_e''(j\omega) \mp_e^o \cdots.\end{aligned}$$

Thus the Lemma 2 can be restated as follows:

Lemma 3: $G(s)$ is SPR if and only if:

1) $G(s)$ is analytic in $\mathrm{Re}[s] \geq 0$,
2) $n - m \geq -1$ and if $n - m = -1$ then $k > 0$,
3) For each $\omega \geq 0$ inequality:

$$\lim_{\varepsilon\to 0^+}\left[G_e(j\omega) - \varepsilon G_o'(j\omega) + \frac{1}{2!}\varepsilon^2 G_e''(j\omega) \mp_e^o \ldots\right] \geq 0.$$

In the next section these results are used to study the previous well-known common statements in the SPR transfer function literatures.

4. PREVIOUS WELL-KNOWN COMMON STATEMENTS

In spite of the basic definition of SPR functions which has been motivated by Popov hyperstability theory and restated in the frequency domain [3], almost all activities have been focused on the state space approaches based on KYP Lemma. Our purpose here is to verify the strict positive realness of real transfer functions via Lemma 3 which is derived directly from frequency domain definitions. First, four well-known common statements that yield the necessary and sufficient conditions for SPR functions are discussed and then their drawbacks are mentioned through some examples.

The basic question that the four common statements in SPR area try to answer is:

Which extra conditions must be held on a PR transfer function to guarantee it to be an SPR transfer function? The main object of this paper is to clarify the answer of this question.

Statement 1: (Astrom) [4]

Theorem: $G(s)$ is SPR if and only if:

1) $G(s)$ is analytic in $\mathrm{Re}[s] \geq 0$,
2) $G(s)$ has no any pole or zero on the imaginary axis,
3) $\mathrm{Re}[G(j\omega)] \geq 0,\ \forall \omega \geq 0$.

Counter Example 1: According to this statement the transfer function $G_1(s) = \dfrac{s^2+s+1}{s^2+s+4}$ is SPR, because:

$$\mathrm{Re}[G_1(j\omega)] = G_{1e}(j\omega) = \frac{(\omega^2-2)^2}{(\omega^2-4)^2+\omega^2} \geq 0,\ \forall \omega \geq 0.$$

But using Lemma 3, we have:

$$\mathrm{Re}[G_1(j\omega-\varepsilon)] = G_{1e}(s) - \varepsilon G_{1o}'(s) \pm \cdots \Big|_{s=j\omega}$$

$$= \frac{(s^2+2)^2}{(s^2+4)^2-s^2} - \varepsilon\left(\frac{-9s^2-21s^2+48}{\left((s^2+4)^2-s^2\right)^2}\right) \pm \cdots \Bigg|_{s=j\omega}$$

$$= \frac{(\omega^2-2)^2}{(\omega^2-4)^2+\omega^2} - \varepsilon\left(\frac{-9\omega^4+21\omega^2+48}{\left((\omega^2-4)^2+\omega^2\right)^2}\right) \pm \cdots.$$

Now it is easy to verify that

$$\mathrm{Re}[G_1(j\sqrt{2}-\varepsilon)] = -1.5\varepsilon + h.o.t(\varepsilon).$$

Thus G(s) is not SPR by the basic definitions because

$$\lim_{\varepsilon \to 0^+} \mathrm{Re}[G_1(j\sqrt{2}-\varepsilon)] = -1.5\varepsilon < 0.$$

Statement 2: (Slotine) [5]

Theorem: $G(s)$ is SPR, if and only if:

1) $G(s)$ is analytic in $\mathrm{Re}[s] \geq 0$,
2) $\mathrm{Re}[G(j\omega)] > 0,\ \forall \omega \geq 0$.

Counter Example 2: According to this statement the transfer function $G_2(s) = \dfrac{s+1}{s^2+s+1}$ is SPR, because:

$$\mathrm{Re}[G_2(j\omega)] = G_{2e}(j\omega) = \frac{1}{(\omega^2-1)^2+\omega^2} > 0,\ \forall \omega \geq 0.$$

But using Lemma 3, we have:

$$\mathrm{Re}[G_2(j\omega-\varepsilon)] = \frac{1}{(\omega^2-1)^2+\omega^2} - \varepsilon\left(\frac{\omega^6+\omega^4-3\omega^2}{\left((\omega^2-1)^2+\omega^2\right)^2}\right) \pm \cdots.$$

Thus G(s) is not SPR according to the basic definitions because:

$$\mathrm{Re}_{\omega \to \infty}[G_2(j\omega-\varepsilon)] \approx -\frac{\varepsilon}{\omega^2} < 0,\ \forall \varepsilon > 0.$$

Comment 1: It should be noted that, if r is the relative degree of $G(s)$, then the relative degree of $G^{(k)}(s)$ is $(r+k)$, hence the first two terms of the Taylor series are sufficient to study the behavior of $G(s)$ in sufficiently large frequencies as used in previous counter example.

Statement 3: (Ioannou and Tao) [6]

Theorem: $G(s)$ is SPR, if and only if:

1) $G(s)$ is analytic in $\mathrm{Re}[s] \geq 0$,
2) $\mathrm{Re}[G(j\omega)] > 0,\ \forall \omega \geq 0$,
3) One of the following conditions is satisfied:

a) If $n-m=-1$ Then: $\begin{cases} i)\ \lim_{\omega \to \infty} \mathrm{Re}[G(j\omega)] > 0 \\ ii)\ \lim_{s \to \infty} \dfrac{G(s)}{s} > 0, \end{cases}$

b) If $n-m=1$, then $\lim_{\omega \to \infty} \omega^2\, \mathrm{Re}[G(j\omega)] > 0$.

Comment 2: Since Statement 3 has been proved in the state space, it doesn't involve improper transfer functions. It should be noted that the condition (3-a-i) is not appeared in [8,9], so the necessity of this condition is mentioned by following example.

Example 1: Suppose $G_3(s) = \dfrac{s^2+s+1}{s+1}$. It is easy to show that all conditions in theorem 3 are satisfied for G_3(s) except the condition (3-a-i) because:

$$\lim_{\omega \to \infty} \mathrm{Re}[G_3(j\omega)] = \lim_{\omega \to \infty} \frac{1}{1+\omega^2} = 0.$$

But according to Lemma 3 we have:

$$\mathrm{Re}[G_3(j\omega-\varepsilon)] = \frac{1}{\omega^2+1} - \varepsilon\left(\frac{\omega^4+3\omega^2}{(\omega^2+1)^2}\right) \pm \cdots.$$

So $\mathrm{Re}_{\omega \to \infty}[G_3(j\omega-\varepsilon)] \approx -\varepsilon < 0$, and thus G_3(s) is not SPR.

Statement 4: (Khalil) [7]

Theorem: Suppose $G(s)$ denotes a proper rational transfer function, then $G(s)$ is SPR, if and only if:

1) $G(s)$ is analytic in $\mathrm{Re}[s] \geq 0$,
2) $\mathrm{Re}[G(j\omega)] > 0,\ \forall \omega \geq 0$,
3) One of the following conditions is satisfied:

a) If $G(\infty)=0$ (i.e., $n-m=0$) then $k>0$,

b) If $n-m=1$ then, $\lim_{\omega\to\infty}\omega^2\operatorname{Re}[G(j\omega)]>0$.

This theorem considers only proper functions due to the restriction arises from state space realization and KYP Lemma. In general, the third condition which is appeared in the two last theorems refers to the fact that there are extra necessary conditions which are imposed by high frequency behavior of some specific functions (i.e., $\operatorname{Re}[G(j\omega)]$, $G(s)/s$ and $\omega^2\operatorname{Re}[G(j\omega)]$. In the next section these necessary conditions is substituted by a new simpler necessary condition which is resulted from the Nyquist diagram of $G(s)$ over the range of high frequencies.

5. MAIN RESULTS

Suppose $G(s)$ is a real rational transfer function as shown in (1):

$$G(s)=k\frac{b(s)}{a(s)}=k\frac{s^m+b_1s^{m-1}+\ldots+b_m}{s^n+a_1s^{n-1}+\ldots+a_n},\ k>0.$$

It should be noted that the inequalities $|n-m|\le 1$, $k>0$ and $b_i\ge 0$, $i=1,\ldots,m$; $a_j\ge 0$, $j=1,\ldots,n$ can easily be resulted from circuit theory for any positive real function. In the case $k>0$, if we assume $n-m=0$, then there is no need for any extra condition which is imposed by high frequencies for $G(s)$ to be strict positive real, because $G(s)\to k$ as $s\to\infty$.

For the case $|n-m|=1,\ k>0$, the new condition $(n-m)(a_1-b_1)>0$ must be satisfied for $G(s)$ to be strict positive real. This fact is proved in the following proposed Lemma.

Lemma 4: Suppose $G(s)$ is a SPR real rational transfer function as in (1) and $|n-m|=1$, then $(n-m)(a_1-b_1)>0$.

Proof: For $|n-m|=1$, it is obvious that if $G(s)$ is PR then its Nyquist diagram lies on the closed right half $G(s)$-plane and converges to origin when frequency goes to infinity, so the derivative of $\arg G(j\omega)$ is non-positive at sufficiently high frequencies. In the other words and as it is shown in Fig.1 Only diagrams ❶ and ❷ may belong to the set of PR transfer functions with relative degree one. The diagram ❷ shows the Nyquist diagram of an odd transfer function, i.e., $G(s)=-G(-s)$, which it's all poles and zeros are on the imaginary axis and thus is not SPR. The diagram ❶ shows that the derivative of $\arg G(j\omega)$ is non-positive at sufficiently high frequencies.

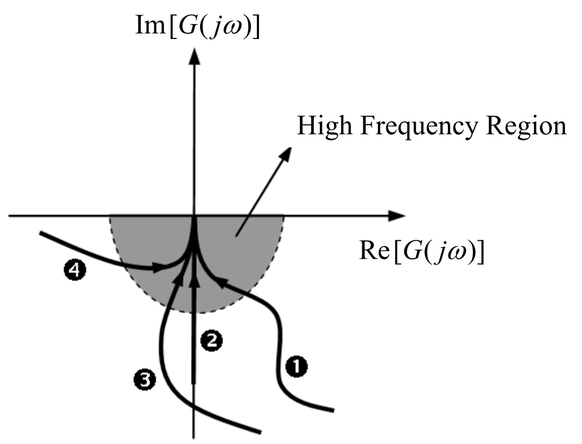

Fig. 1. Nyquist diagram of different $G(s)$'s with relative degree one (n-m=1).

Also the equality $\arg\{G(j\omega)\}=-\arg G(j\omega)$, implies that when $n-m=-1$, the derivative of $\arg G(j\omega)$ is positive at sufficiently high frequencies. Hence if $G(s)$ is PR and $|n-m|=1$ then:

$$(n-m)\left(\frac{d}{d\omega}\arg G(j\omega)\right)\le 0,\ |n-m|=1. \qquad (3)$$
$$\omega\to\infty$$

We know if $G(s)$ is SPR then $G(s-\varepsilon)$ is PR, therefore if $G(s)$ is SPR and $|n-m|=1$ then:

$$(n-m)\left(\frac{d}{d\omega}\arg G(j\omega-\varepsilon)\right)\le 0,\ |n-m|=1.$$
$$\omega\to\infty$$

Now by substituting $s=j\omega-\varepsilon$ in (1) we have:

$$G(j\omega-\varepsilon)=k\frac{(j\omega-\varepsilon)^m+b_1(j\omega-\varepsilon)^{m-1}\ldots+b_m}{(j\omega-\varepsilon)^n+a_1(j\omega-\varepsilon)^{n-1}\ldots+a_n}$$
$$=k\frac{\prod_{i=1}^{m}\left(j\omega-\varepsilon-z_i\right)}{\prod_{l=1}^{n}\left(j\omega-\varepsilon-p_l\right)}. \qquad (4)$$

It is easy to see that

$$\arg G(j\omega-\varepsilon)=\sum_{i=1}^{m}\tan^{-1}\left(\frac{\omega-\operatorname{Im}z_i}{-\varepsilon-\operatorname{Re}z_i}\right)-\sum_{l=1}^{n}\tan^{-1}\left(\frac{\omega-\operatorname{Im}p_l}{-\varepsilon-\operatorname{Re}p_l}\right). \qquad (5)$$

Therefore

$$\frac{d}{d\omega}\arg G(j\omega-\varepsilon)$$
$$=\sum_{i=1}^{m}\left(\frac{-\varepsilon-\operatorname{Re} z_i}{\left(\varepsilon+\operatorname{Re} z_i\right)^2+\left(\omega-\operatorname{Im} z_i\right)^2}\right)$$
$$-\sum_{l=1}^{n}\left(\frac{-\varepsilon-\operatorname{Re} p_l}{\left(\varepsilon+\operatorname{Re} p_l\right)^2+\left(\omega-\operatorname{Im} p_l\right)^2}\right). \tag{6}$$

If ω tends to infinity we have:

$$\frac{d}{d\omega}\arg G(j\omega-\varepsilon)\underset{\omega\to\infty}{\approx}\sum_{i=1}^{m}\left(\frac{-\varepsilon-\operatorname{Re} z_i}{\omega^2}\right)$$
$$-\sum_{l=1}^{n}\left(\frac{-\varepsilon-\operatorname{Re} p_l}{\omega^2}\right)=\frac{b_1-a_1+(n-m)\varepsilon}{\omega^2}. \tag{7}$$

Multiplying both sides of (7) by (n-m) and noting to $|n-m|=1,$ we have:

$$(n-m)\left(\frac{d}{d\omega}\arg G(j\omega-\varepsilon)\right)\underset{\omega\to\infty}{=}$$
$$\frac{(n-m)\left(b_1-a_1\right)+(n-m)^2\varepsilon}{\omega^2}=\frac{(n-m)\left(b_1-a_1\right)+\varepsilon}{\omega^2}.$$

Therefore, the inequality (3) implies $(n-m)$ $(a_1-b_1)\ge\varepsilon>0,$ so the proof is completed. □

Remark 1: If $G(s)$ is PR and $|n-m|=1,$ then the inequality $(n-m)(a_1-b_1)\ge 0$ will be satisfied.

Remark 2: If $G(s)$ is SPR and $|n-m|=1,$ then (7) and Lemma 4 show that $\frac{d}{d\omega}\arg G(j\omega)$ doesn't decay more rapidly than ω^{-2} as $|\omega|\to\infty.$

Comment 3: The restriction which is introduced in Remark 2 does not exist for the PR functions. This restriction is an important difference between PR and SPR functions related to high frequency behavior of the Nyquist diagram of $G(s)$. In the other words, the third condition which is appeared in Statements 3 and 4 can be replaced by: if $|n-m|=1,$ then $a_1\neq b_1$. It is clear that $-b_1$ is equal to the summation of the zeros of $G(s)$ and $-a_1$ is equal to the summation of the poles of $G(s)$. Hence the new necessary condition which is stated in Lemma 4 for $G(s)$ with relative degree one, can be interpreted as:

$$a_1-b_1=\left(\sum_{l=1}^{n}\operatorname{Re}[-p_l]\right)-\left(\sum_{i=1}^{m}\operatorname{Re}[-z_i]\right)>0. \tag{8}$$

Comment 4: Suppose $G(s)$ is in the form of (1) and has the relative degree one, then

$$\operatorname{Re}[G(j\omega)]=k\frac{\operatorname{Re}\{b(j\omega)a(-j\omega)\}}{a(j\omega)a(-j\omega)}$$
$$=k\frac{\left(a_1-b_1\right)\omega^{2(n-1)}+\ldots}{\omega^{2n}+\ldots} \tag{9}$$

ensures that

$$\lim_{\omega\to\infty}\omega^2\operatorname{Re}[G(j\omega)]=k(a_1-b_1).$$

Thus the condition:

If $n-m=1$ then $\lim_{\omega\to\infty}\omega^2\operatorname{Re}[G(j\omega)]>0$

appeared in Statements 3 and 4 can be restated as:

If $n-m=1$ then $k(a_1-b_1)>0.$

Since the second condition in Statements 3 and 4 ensures that $k(a_1-b_1)\ge 0$ for $n-m=1,$ therefore this condition can be simplified as:

If $n-m=1$ then $a_1\neq b_1.$

The following examples illustrate the use of the proposed necessary condition.

Example 2: Let

$$G_4(s)=\frac{(s+4)(s+6)}{(s+2)(s+3)(s+5)},$$

we have $a_1-b_1=(2+3+5)-(4+6)=0,$ Thus according to the proposed lemma (Lemma 4) and Remark 1, $G_4(s)$ is not SPR but maybe PR. Fig. 2 shows that $G_4(s)$ is PR.

Example 3: Let

$$G_5(s)=\frac{s^2+s+1}{2s^3+2s^2+3s+2},$$

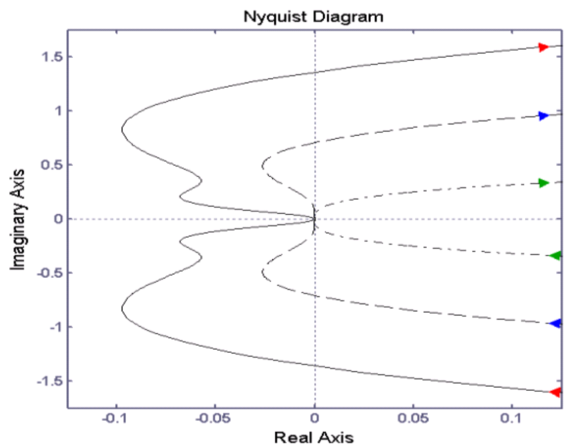

Fig. 2. $G_4(s)$: _._._ , $G_5(s)$: __ __ , $G_6(s)$: _____.

we have $a_1 - b_1 = (2/2) - (1) = 0$. Thus according to Lemma 4 and Remark 1, $G_5(s)$ is not SPR but maybe PR. Fig. 2 shows that $G_5(s)$ is not PR.

Example 4: Let

$$G_6(s) = \frac{(s^2+6s+11)(s^2+5s+3)}{(s^2+3s+7)(3s^3+18s^2+5s+9)},$$

we have $a_1 - b_1 = (3+18/3) - (6+5) = -2$.

Thus according to Lemma 4 and Remark 1, $G_6(s)$ is not PR.

Fig. 2 shows that $G_6(s)$ is not PR.

Theorem 1: $G(s)$ is SPR, if and only if:

1) $G(s)$ is analytic in $\mathrm{Re}[s] \geq 0$,
2) $n - m \geq 1$, and moreover if $n - m = 1$, then $k > 0$,
3) $\mathrm{Re}[G(j\omega)] > 0, \forall \omega \geq 0$,
4) If the relative degree of $G(s)$ is nonzero then the summation of zeros of $G(s)$ $(-b_1)$ must not be equal to the summation of poles of $G(s)$ $(-a_1)$ i.e., if $|n-m| = 1$ then $a_1 \neq b_1$.

Proof: An important result which follows from the Maximum Modulus Principle can be stated as follows: Suppose $G(s)$ is a function of complex variable $s = \sigma + j\omega$, now if it is analytic in a closed bounded region Γ and is not constant throughout the interior of Γ, then $\mathrm{Re}[G(s)]$ has a minimum value in Γ which occurs on the boundary of Γ and never in the interior of Γ [30].

Now considering Lemma 1, the first condition states that $G(s)$ is analytic in $\mathrm{Re}[s] \geq 0$, therefore the minimum value of Re $[G(s)]$ occurs on the boundary $s = -\varepsilon + j\omega, \forall \omega \in R$ which is stated in the third condition of Lemma 1. If $\mathrm{Re}[G(j\omega)] \geq 0$ then there is a finite frequency ω_0 such that $\mathrm{Re}[G(j\omega_0)] = 0$ and the above result of Maximum Modulus Principle implies that $\mathrm{Re}[G(j\omega_0 - \varepsilon)] < 0, \forall \varepsilon > 0$ and thus the inequality $\mathrm{Re}[G(j\omega)] > 0$ is necessary for $\mathrm{Re}_{\varepsilon \to 0}$ $+[G(j\omega - \varepsilon)] \geq 0, \forall \omega \in R$. The fourth condition can be proved by considering the inequality $\mathrm{Re}[G(j\omega - \varepsilon)] \geq 0$, $\forall \varepsilon > 0$ at the sufficiently large frequencies as discussed in the previous section. □

Comment 5: It follows from circuit theory that for any positive real function the inequalities $k > 0$, $|n-m| \leq 1$ must be satisfied. Thus the theorem (1) can be restated as follows:

Theorem 2: The real rational transfer function

$$G(s) = k\frac{b(s)}{a(s)} = k\frac{s^m + b_1 s^{m-1} + \ldots + b_m}{s^n + a_1 s^{n-1} + \ldots + a_n}, \quad k \neq 0,$$

is SPR, if and only if:

1) $|n-m| \leq 1$ and $k > 0$, and moreover if $|n-m| = 1$ then $a_1 \neq b_1$,
2) $G(s)$ is analytic in $\mathrm{Re}[s] \geq 0$,
3) $\mathrm{Re}[G(j\omega)] > 0, \forall \omega \geq 0$.

Comment 6: If G(s) is PR, then it will be SPR if the following extra conditions are satisfied:

1) If $|n-m| = 1$ then $a_1 \neq b_1$,
2) $G(s)$ has no pole or zero on the imaginary axis,
3) $\mathrm{Re}[G(j\omega)] \neq 0, \forall \omega \in R$.

6. CONCLUSION

In this paper, unlike the other works in strict positive realness area which have focused on the state space tools such as KYP Lemma, the necessary and sufficient conditions for SPR functions are derived directly from the basic definitions in the frequency domain using complex analysis tools. The proposed approach is established based on the Taylor expansion and the Maximum Modulus Principle which are the fundamental tools in the complex analysis. Using Taylor expansion approach, the four common statements in the strict positive realness area investigated. A new necessary condition based on the high frequency behavior of the transfer function is extracted. Finally through a new theorem a more simplified and completed conditions for strict positive realness in frequency domain are presented.

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