

Kalman Filtering for Linear Time-Delayed Continuous-Time Systems with Stochastic Multiplicative Noises

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Abstract: The paper deals with the Kalman stochastic filtering problem for linear continuous-time systems with both instantaneous and time-delayed measurements. Different from the standard linear system, the system state is corrupted by multiplicative white noise, and the instantaneous measurement and the delayed measurement are also corrupted by multiplicative white noise. A new approach to the problem is presented by using projection formulation and *re-organized innovation analysis*. More importantly, the proposed approach in the paper can be applied to solve many complicated problems such as stochastic H_∞ estimation, H_∞ control stochastic system with preview and so on.

Keywords: Delay systems, filtering, innovation analysis, Riccati differential equations.

1. INTRODUCTION

In recent years, stochastic filtering has become a popular research topic, and it has been extensively developed [6,8,14,20,21]. The stochastic filtering problem as well as the stochastic control problem have wide application in areas such as nuclear fission and heat transfer, population models and immunology, see [2,5] and references therein. There are mainly two performances under which the filtering problem has been considered, one is the H_2 index, [10,12,13,16], and the other one is H_∞ performance, see [6,8,9,14,15]. The H_2 filtering which is also termed as Kalman filtering is to address the minimization of the filtering error covariance. The so-called H_∞ stochastic filtering is to design an estimator to estimate the unknown state combination via output measurement, which guarantees the L_2 gain (from the external disturbance to the estimation error) to be less than a prescribed level $\gamma > 0$.

[13] presented a robust recursive Kalman filtering algorithm that addresses estimation problems arising in linear time-varying systems with stochastic

parametric uncertainties. In [8], the discrete-time H_∞ stochastic filtering is considered and a filter of Luenberger-type structure is given. A modified-Riccati recursion which guarantees a given H_∞ filtering level to minimize an upper-bound on the covariance of the estimation error is presented. However, dimension of the resulting modified Riccati equation is larger than the origin system. In [15], the stationary discrete-time linear stochastic filtering is considered, where the stochastic uncertainty appears in the dynamic, input and measurement matrices. A filter which is based on the stochastic bounded real lemma (BRL) of [9] is given in terms of LMIs. However, the condition is only sufficient. In [14], continuous-time stochastic linear plants which are controlled by dynamic output feedback and subjected to both deterministic and stochastic perturbations are considered. A bounded real lemma is given and the necessary and sufficient conditions for the existence of a stabilizing compensator were obtained. Based on the BRL proposed in [14], [4] gives the continuous-time H_∞ stochastic filter, and the sufficient and necessary condition for the existence of the solutions in terms of LMIs is presented.

The above mentioned works are normal systems without delay. For the case of time-delay, retarded type linear systems robust L_2 filtering problem was considered in [19], where delay-independent conditions were derived in the form of LMIs. A more general neutral type of systems and a less conservative delay-dependent filter is given in [11]. Robust H_∞ filtering of uncertain systems with state delays has been considered in [17,18], only sufficient conditions have been derived. For the above mentioned works, the systems aren't corrupted by multiple white noise and delayless measurement can be used.

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In this paper, we consider the finite horizon robust Kalman stochastic filtering problem for linear time-varying continuous-time system with delayed measurement, assuming that the system state is corrupted by white noise, and the measurement including the delayed measurement are also corrupted by white noise. Different from the works for the robust stochastic filter [13], delayed measurement is introduced in this paper. By using re-organized innovation analysis based on the knowledge of Krein space [1,21-24], a new approach to the Kalman stochastic filtering is proposed. With the proposed new approach, the stochastic filter for time delayed system is obtained. The solution is given in terms of Riccati equation which has the same dimension as the origin system.

The rest of the paper is organized as follows. The system under consideration and the problem statement are given in Section 2. Main results concluding the Riccati equation and the optimal stochastic filter are presented in Section 3. Some concluding remarks are made in Section 4 as sensor fusion [7] and network systems. The key is the re-organization of the measurements and innovation.

2. PROBLEM STATEMENT

Consider the following stochastic differential systems

$$\dot{x}(t) = A(t)x(t) + B(t)x(t)w(t) + u(t), \quad (1)$$

$$y_0(t) = C_0(t)x(t) + D_0(t)x(t)w(t) + v_0(t), \quad (2)$$

$$y_1(t) = C_1(t)x(t) + D_1(t)x(t)w(t) + v_1(t), \quad (3)$$

where $t_l = t - l, t \leq T$, with T is the horizon, $x(t) \in R^n$ is the state, $y_0(t) \in R^{p_0}$ is measurement, and $y_1(t) \in R^{p_1}$ is time-delayed measurements. $u(t) \in R^n$ is input sequence, $w(t) \in R^1$ $v_i(t) \in R^{p_i}$ are white noises of zero means. $A(t) \in R^{n \times n}$, $B(t) \in R^{n \times n}$, $C_i(t) \in R^{p_i \times n}$ and $D_i(t) \in R^{p_i \times n}$ are known time-varying matrices. $x(t)|_{t=0} = x_0$ and $\langle u(t), u(s) \rangle = Q(t)\delta_{t,s}$, $\langle v_i(t), v_j(t) \rangle = R_i(t)\delta_{i,j}\delta_{t,s}$, $\langle w(t), w(s) \rangle = M(t)\delta_{t,s}$ and $\langle x_0, x_0 \rangle = \prod 0$.

The stochastic filtering problem considered in the paper for the system model (1)-(3) can be stated as

Problem P: Given the observation $\{y_0(\tau)|_{0 \leq \tau \leq t}, y_1(\tau)|_{l \leq \tau \leq t}\}$, find the optimal estimation of $x(t)$, denoted $\hat{x}(t|t)$, such that

$$\min \mathcal{E} \left\{ \left[x(t) - \hat{x}(t|t) \right]' \left[x(t) - \hat{x}(t|t) \right] \right\}, \quad (4)$$

where \mathcal{E} denotes the mathematical expectation.

Remark 1: The stochastic Kalman filtering problem for discrete-time system has been considered in [13], where delayless measurement is available. However, in practice, it is inevitable that delay exists in system, then we give the system model (1)-(3). Such problem has important applications in many engineering problems such as sensor fusion [7] and network systems. The key is the re-organization of the measurements and innovation.

Let $y(t)$ denote the observation of the system (4) at time t and $v(t)$ is the related observation noise at time t then we have

$$y(t) = \begin{cases} y_0(t), & 0 \leq t \leq l \\ \begin{bmatrix} y_0(t) \\ y_1(t) \end{bmatrix}, & t \geq l, \end{cases} \quad (5)$$

$$v(t) = \begin{cases} v_0(t), & 0 \leq t \leq l \\ \begin{bmatrix} v_0(t) \\ v_1(t) \end{bmatrix}, & t \geq l. \end{cases} \quad (6)$$

Remark 2: For $t < l$, the measurement is mainly from one channel $y_0(t)$, and when $t \geq l$, the measurement is two channel $y_0(t)$ and $y_1(t)$.

Remark 3: In our paper, the main idea is to reorganize the delayed measurements (that is $y_0(t)$, $y_1(t)$) into delay-free measurements from different systems ($y(t)$), thus (5) and (6) can be easily introduced. In this way, the system with delayed measurement has been changed into a system with two different measurements without delay.

For the convenience of discussion, we will consider the case of $t \geq l$, the case of $t < l$ can be studied in the same way.

3. MAIN RESULTS

In this section, we will introduce the re-organized innovation in terms of which, the new stochastic filter will be given [22].

3.1. Re-organized measurements

As is well known, given the measurement sequence $\{y(s)|_{0 \leq s \leq t}\}$, the optimal state filter $\hat{x}(t|t)$ is the projection of $x(t)$ onto the linear space spanned by the measurement sequence, denoted by $\mathcal{L}\{y(s)|_{0 \leq s \leq t}\}$ [3]. In the following part, we shall re-organize the measurement sequence and give a new observation sequence which is delayless.

Lemma 1: The linear space of $\mathcal{L}\{y(s)|_{0 \leq s \leq t}\}$ is equivalent to

$$\mathcal{L}\left\{\mathcal{Y}_2(s)\Big|_{0\leq s\leq t}; \mathcal{Y}_1(s)\Big|_{t_l\leq s\leq t}\right\}, \quad (7)$$

where

$$\mathcal{Y}_2(s) = \begin{bmatrix} y_0(s) \\ y_1(s+l) \end{bmatrix}, \quad (8)$$

$$\mathcal{Y}_1(s) = y_0(s)$$

satisfies that

$$\mathcal{Y}_1(s) = \mathcal{E}_1(s)x(s) + \mathcal{D}_1(s)x(s)w(s) + \mathcal{V}_1(s), \quad (9)$$

$$\mathcal{Y}_2(s) = \mathcal{E}_2(s)x(s) + \mathcal{D}_2(s)x(s)w(s) + \mathcal{V}_2(s) \quad (10)$$

with

$$\mathcal{E}_2(s) = \begin{bmatrix} C_0(s) \\ C_1(s+l) \end{bmatrix}, \mathcal{D}_2(s) = \begin{bmatrix} D_0(s)w(s) \\ D_1(s)w(s+l) \end{bmatrix},$$

$$\mathcal{V}_2(s) = \begin{bmatrix} v_0(s) \\ v_1(s+l) \end{bmatrix}, \mathcal{E}_1(s) = C_0(s), \quad (11)$$

$$\mathcal{D}_1(s) = D_0(s), \mathcal{V}_1(s) = v_0(s).$$

Moreover, $\mathcal{V}_i(s)$ is a white noise of zero means and covariance matrices

$$Q_{\mathcal{V}_2}(s) = \text{diag}\{R_0(s), R_1(s+l)\}, \quad (12)$$

$$Q_{\mathcal{V}_1}(s) = R_0(s).$$

Proof: It is straightforward by re-organizing the delayed observations.

Remark 4: In the above lemma, by re-organizing all the measurements, the system with l -delay measurements has been changed into a normal system with measurement of (9) which does not involve delays.

3.2. Re-organized innovation sequence

Before presenting the re-organized innovation, we will give the following definitions first.

Definition 1: Given time instant t , the estimator $\hat{\xi}(s,1)$ for $s > t_l$ is the optimal estimation of $\xi(s)$ given the observation

$$\mathcal{L}\left\{\mathcal{Y}_2(\tau)\Big|_{0\leq\tau\leq t_l}; \mathcal{Y}_1(\tau)\Big|_{t_l\leq\tau\leq s}\right\}. \quad (13)$$

For $s = t_l$, $\hat{\xi}(s,1)$ is the optimal estimation of $\xi(s)$ given the observation

$$\mathcal{L}\left\{\mathcal{Y}_2(\tau)\Big|_{0\leq\tau\leq t_l}\right\}. \quad (14)$$

Definition 2: Given time instant t , the estimator $\hat{\xi}(s,2)$ for $s < t_l$, $\hat{\xi}(s,2)$ is the optimal estimation of $\xi(s)$ given the observation

$$\mathcal{L}\left\{\mathcal{Y}_2(\tau)\Big|_{0\leq\tau\leq s}\right\}. \quad (15)$$

Remark 5: ξ in $\hat{\xi}(s,i)$ ($i=1,2$) in the above two definitions may be x or $\mathcal{X}_1, \mathcal{X}_2$.

Definition 3: For $i=2$ and any $s \geq 0$

$$\mathcal{W}_2(s) \triangleq \mathcal{Y}_2(s) - \hat{\mathcal{Y}}_2(s,2), \quad (16)$$

and for $i=1$ and any $s > 0$

$$\mathcal{W}_1(t_l+s) \triangleq \mathcal{Y}_1(t_l+s) - \hat{\mathcal{Y}}_1(t_l+s,1), \quad (17)$$

where $\hat{\mathcal{Y}}_i(s,i)$ is as defined in Definitions 2 and 3.

Thus we have the following lemma.

Lemma 2: The sequence \mathcal{W} defined in the above is mutually uncorrelated and

$$\left\{\mathcal{W}_2(s)\Big|_{0\leq s\leq l}; \mathcal{W}_1(t+s)\Big|_{0\leq s\leq l}\right\}, \quad (18)$$

spans the same linear space as $\mathcal{L}\{y(s)\Big|_{0\leq s\leq t}\}$.

Proof: By recalling [22,23] where the case for the linear system without multiplicative noise is considered, the proof can be established by applying a similar argument as in [22,23].

Remark 6: It will be seen in the following discussions, the re-organized innovation presented in the subsection will play an important role in deriving the Kalman filter for the system with measurement delays.

From the definition (16)-(17), it is easy to observe the relationships as

$$\mathcal{W}_2(s) = \mathcal{E}_2(s)e(s) + \mathcal{D}_2(s)x(s) + \mathcal{V}_2(s), \quad (19)$$

$$\begin{aligned} \mathcal{W}_2(t_l+s) &= \mathcal{E}_1(t_l+s)e_1(t_l+s) \\ &+ \mathcal{D}_1(t_l+s)x(t_l+s)w(t_l+s) \\ &+ \mathcal{V}_1(t_l+s), s > 0, \end{aligned} \quad (20)$$

where

$$e_2(s) = x(s) - \hat{x}(s,2), \quad (21)$$

$$e_1(t_l+s) = x(t_l+s) - \hat{x}(t_l+s,1). \quad (22)$$

As in the standard Kalman filtering formulation, $\mathcal{E}_2(s)$ and $\mathcal{E}_1(t_l+s)$ play key role for computing the filter for the system with measurement delays, it is obvious

$$\mathcal{E}_2(s) \triangleq \langle e_2(s), e_2(s) \rangle, s \geq 0, \quad (23)$$

$$\mathcal{E}_1(t_l+s) \triangleq \langle e_1(t_l+s), e_1(t_l+s) \rangle, \quad (24)$$

$e_2(s)$ and $e_1(t_l+s)$ are as in (21) and (22), respectively. $\Pi(s)$ is the state covariance matrix and is defined as

$$\Pi(s) \triangleq \langle x(s), x(s) \rangle, s \geq 0. \quad (25)$$

Then we have the following theorem for computing $\mathcal{E}_2(s)$ and $\mathcal{E}_1(t_l+s)$.

Theorem 1: The one-step state predictive covariance matrices defined in the above are calculated by:

1) $\mathcal{P}_2(s)$ is the solution to the following Riccati differential equation

$$\begin{aligned} \frac{d\mathcal{P}_2(s)}{ds} &= A(s)\mathcal{P}_2(s) + \mathcal{P}_2(s)A'(s) \\ &+ B(s)\prod(s)B'(s)M(s) \\ &- \mathcal{K}_2(s)Q_{w_2}(s)\mathcal{K}_2'(s) \\ &+ Q(s), \mathcal{P}_2(0) = \prod_0, \end{aligned} \quad (26)$$

$$\begin{aligned} \frac{d\prod(s)}{ds} &= A(s)\prod(s) + \prod(s)A'(s) \\ &+ B(s)\prod(s)B'(s)M(s) \\ &+ Q(s), \prod(s) = \prod_0, \end{aligned} \quad (27)$$

where

$$\begin{aligned} \mathcal{K}_2(s) &= [\mathcal{P}_2(s)C_0'(s) + B(s)\prod(s)D_0'(s) \\ &+ \mathcal{P}_2(s)C_1'(s+l)] \times Q_{w_2}^{-1}(s) \end{aligned} \quad (28)$$

and

$$\begin{aligned} Q_{w_2}(s) &= \text{diag}\{D_0(s)\prod(s)M(s)D_0'(s) + R_0(s), \\ &D_1(s+l)\prod(s)M(s+l)D_1'(s+l) + R_1(s+l)\}. \end{aligned} \quad (29)$$

2) $\mathcal{A}(t_l + s)$ is the solution to the following Riccati differential equation

$$\begin{aligned} \frac{d\mathcal{A}(t_l + s)}{ds} &= A(t_l + s)\mathcal{A}(t_l + s) + \mathcal{A}(t_l + s)A'(t_l + s) \\ &+ B(t_l + s)\prod(t_l + s)B'(t_l + s)M(t_l + s) \\ &- \mathcal{K}_1(t_l + s)Q_{w_1}(t_l + s)\mathcal{K}_1'(t_l + s) \\ &+ Q(t_l + s), \mathcal{A}(t_l) = \mathcal{P}_2(t_l), \end{aligned} \quad (30)$$

$$\begin{aligned} \frac{d\prod(t_l + s)}{ds} &= A(t_l + s)\prod(t_l + s) + \prod(t_l + s)A'(t_l + s) \\ &+ B(t_l + s)\prod(t_l + s)B'(t_l + s)M(t_l + s) \\ &+ Q(t_l + s), \end{aligned} \quad (31)$$

where

$$\begin{aligned} \mathcal{K}_1(t_l + s) &= [\mathcal{A}(t_l + s)\mathcal{E}_1'(t_l + s) + B(t_l + s) \\ &\times \prod(t_l + s)\mathcal{D}_1'(t_l + s)M(t_l + s)] \\ &\times Q_{w_1}^{-1}(t_l + s) \end{aligned} \quad (32)$$

with

$$\begin{aligned} Q_{w_1}(t_l + s) &= \mathcal{E}_1(t_l + s)\prod(t_l + s)M(t_l + s)\mathcal{E}_1'(t_l + s) \\ &+ Q_v(t_l + s). \end{aligned} \quad (33)$$

Proof: 1) The proof of (26)-(28):

By using Lemma 1 state-space model (1)-(3) can be approximated as

$$\begin{aligned} \frac{x(i\Delta + \Delta) - x(i\Delta)}{\Delta} &= A(i\Delta)x(i\Delta) + u(i\Delta) \\ &+ B(i\Delta)x(i\Delta)w(i\Delta), \end{aligned} \quad (34)$$

$$\mathcal{Z}_2(i\Delta) = \mathcal{E}_2(i\Delta)x(i\Delta) + \mathcal{D}_2(i\Delta)x(i\Delta) + \mathcal{V}_2(i\Delta), \quad (35)$$

where

$$\begin{aligned} \mathcal{E}_2(i\Delta) &= \begin{bmatrix} C_0(i\Delta) \\ C_1(i\Delta + \frac{l}{t}i\Delta) \end{bmatrix}, \\ \mathcal{D}_2(i\Delta) &= \begin{bmatrix} D_0(i\Delta)w(i\Delta) \\ D_0(i\Delta + \frac{l}{t}i\Delta)w(i\Delta + l\Delta) \end{bmatrix}, \\ \mathcal{V}_2(i\Delta) &= \begin{bmatrix} v_0(i\Delta) \\ v_1(i\Delta + \frac{l}{t}i\Delta) \end{bmatrix}. \end{aligned} \quad (36)$$

Equivalently, (34) and (35) can be rewritten as

$$\begin{aligned} x(i\Delta + \Delta) &= [I + A(i\Delta)\Delta]x(i\Delta) \\ &+ \Delta[B(i\Delta)x(i\Delta)w(i\Delta) + u(i\Delta)], \end{aligned} \quad (37)$$

$$\mathcal{Z}_2(i\Delta) = \mathcal{E}_2(i\Delta)x(i\Delta) + \mathcal{D}_2(i\Delta)x(i\Delta) + \mathcal{V}_2(i\Delta) \quad (38)$$

with

$$\begin{aligned} \langle u(i\Delta), u(i\Delta) \rangle &= \frac{Q(i\Delta)}{\Delta} \delta_{i,j}, \\ \langle \mathcal{V}_2(i\Delta), \mathcal{V}_2(i\Delta) \rangle &= \frac{Q_{\mathcal{V}_2}(i\Delta)}{\Delta} \delta_{i,j}, \\ \langle w(i\Delta), w(i\Delta) \rangle &= \frac{M(i\Delta)}{\Delta} \delta_{i,j}, \end{aligned}$$

and $\langle x_0, x_0 \rangle = \prod_0$.

Applying discrete-time Kalman filtering formulation to the above system, it follows that

$$\begin{aligned} \hat{x}(i\Delta + \Delta, 2) &= [I + A(i\Delta)\Delta]\hat{x}(i\Delta, 2) \\ &+ \mathcal{K}_{p_2}(i\Delta)\mathcal{Z}_2(i\Delta), \end{aligned} \quad (39)$$

$$\hat{x}(0, 2) = 0,$$

where $\mathcal{Z}_2(i\Delta)$ is the innovation of process $\mathcal{Z}_2(i\Delta)$ which is given by

$$\begin{aligned} \mathcal{Z}_2(i\Delta) &\triangleq \mathcal{Z}_2(i\Delta) - \hat{\mathcal{Z}}_2(i\Delta, 2) \\ &= \mathcal{Z}_2(i\Delta) - \mathcal{E}_2(i\Delta)\hat{x}(i\Delta, 2) \\ &= \mathcal{E}_2(i\Delta)e_2(i\Delta) + \mathcal{D}_2(i\Delta)x(i\Delta) \\ &+ \mathcal{V}_2(i\Delta), \end{aligned} \quad (40)$$

where $\hat{\mathcal{Z}}_2(i\Delta, 2)$ and $\hat{x}(i\Delta, 2)$ are respectively the optimal estimation of $\mathcal{Z}_2(i\Delta, 2)$ and $x(i\Delta)$ given

observation $\hat{\mathcal{Z}}_2(s\Delta)(s = i-1, i-2, \dots, 0)$ associated with $u(s)$, $w(s)$ and $\mathcal{Y}_2(s\Delta)$ for $s = i-1, i-2, \dots, 0$, and

$$e_2(i\Delta) \triangleq x(i\Delta) - \hat{x}(i\Delta, 2).$$

Define

$$\Omega(i\Delta) = \text{diag}\{D_0(i\Delta) \prod (i\Delta)M(i\Delta)D_0'(i\Delta), \\ D_1(i\Delta + \frac{l}{t}i\Delta) \prod (i\Delta)M(i\Delta + \frac{l}{t}i\Delta)D_1'(i\Delta + \frac{l}{t}i\Delta)\}$$

then from (40), it can be easily seen

$$\begin{aligned} & \langle \mathcal{Y}_2(i\Delta), \mathcal{Y}_2(i\Delta) \rangle_{u,w,v} \\ &= \left[\mathcal{E}_2(i\Delta)\mathcal{A}_2(i\Delta)\mathcal{E}_2'(i\Delta) + \frac{\Omega(i\Delta)}{\Delta} + \frac{Q_{\mathcal{Y}_2}(i\Delta)}{\Delta} \right] \\ &= \frac{R_{w_2}(i\Delta)}{\Delta}, \end{aligned} \quad (41)$$

that is

$$R_{w_2}(i\Delta) = \Delta \mathcal{E}_2(i\Delta)\mathcal{A}_2(i\Delta)\mathcal{E}_2'(i\Delta) \\ + \Omega(i\Delta) + Q_{\mathcal{Y}_2}(i\Delta), \quad (42)$$

where $\mathcal{A}_2(i\Delta) \triangleq \langle e_2(i\Delta), e_2(i\Delta) \rangle_{u,w,v}$

$$\lim_{\Delta \rightarrow 0, i\Delta \rightarrow s} R(i\Delta) = \Omega(s) + Q_{\mathcal{Y}_2}(s) \triangleq Q_{w_2}(s). \quad (43)$$

That is

$$Q_{w_2}(s) = \text{diag}\{D_0(s) \prod (s)M(s)D_0'(s) \\ + R_0(s), D_1(s+l) \prod (s)M(s+l) \\ \times D_1'(s+l) + R_1(s+l)\}, \quad (44)$$

which is (29).

From (37), (40), and (42), we know

$$\begin{aligned} \mathcal{K}_{p_2}(i\Delta) &= \langle x(i\Delta + \Delta), \mathcal{Y}_2(i\Delta) \rangle \left[\frac{R_{w_2}(i\Delta)}{\Delta} \right]^{-1} \\ &= \langle [I + A(i\Delta)]x(i\Delta), \mathcal{Y}_2(i\Delta) \rangle \left[\frac{R_{w_2}(i\Delta)}{\Delta} \right]^{-1} \\ &+ \langle \Delta B(i\Delta)x(i\Delta)w(i\Delta), \mathcal{Y}_2(i\Delta) \rangle \left[\frac{R_{w_2}(i\Delta)}{\Delta} \right]^{-1} \\ &+ \langle \Delta u(i\Delta), \mathcal{Y}_2(i\Delta) \rangle \left[\frac{R_{w_2}(i\Delta)}{\Delta} \right]^{-1} \\ &= [I + A(i\Delta)\Delta]\mathcal{A}_2(i\Delta)\mathcal{E}_2'(i\Delta) \left[\frac{R_{w_2}(i\Delta)}{\Delta} \right]^{-1} \\ &+ B(i\Delta) \prod (i\Delta) [D_0'(i\Delta)M(i\Delta), 0] \left[\frac{R_{w_2}(i\Delta)}{\Delta} \right]^{-1}, \end{aligned} \quad (45)$$

thus

$$\begin{aligned} \mathcal{K}_2(s) &\triangleq \lim_{\Delta \rightarrow 0} \frac{\mathcal{K}_{p_2}(i\Delta)}{\Delta} \\ &= [\mathcal{K}_{2,11}(s), \mathcal{K}_{2,12}(s)] Q_{w_2}^{-1}(s), \end{aligned} \quad (46)$$

where $\mathcal{K}_{2,11}(s) = \mathcal{A}(s)C_0'(s) + B(s) \prod (s)D_0'(s)M(s)$ and $\mathcal{K}(s) = \mathcal{A}(s)C'(s+l)$.

Combining (40) with (39) yields

$$\begin{aligned} \hat{x}(i\Delta + \Delta, 2) &= [I + A(i\Delta)]\hat{x}(i\Delta, 2) + \mathcal{K}_{p_2}(i\Delta) \\ &\times [\mathcal{Z}_2(i\Delta) - \mathcal{E}_2(i\Delta)\hat{x}(i\Delta, 2)], \end{aligned} \quad (47)$$

which can be rewritten as

$$\begin{aligned} & \frac{\hat{x}(i\Delta + \Delta, 2) - x(i\Delta, 2)}{\Delta} \\ &= A(i\Delta)\hat{x}(i\Delta, 2) \\ &+ \frac{\mathcal{K}_{p_2}(i\Delta)}{\Delta} [\mathcal{Z}_2(i\Delta) - \mathcal{E}_2(i\Delta)\hat{x}(i\Delta, 2)] \end{aligned} \quad (48)$$

$\Delta \rightarrow 0, i\Delta \rightarrow s$ yields

$$\begin{aligned} \frac{d\hat{x}(s, 2)}{ds} &= A(s)\hat{x}(s, 2) \\ &+ \mathcal{K}_2(s)[\mathcal{Z}_2(s) - \mathcal{E}_2(s)\hat{x}(s, 2)]. \end{aligned} \quad (49)$$

Note that

$$\mathcal{Y}_2(i\Delta) = \mathcal{Z}_2(i\Delta) - \mathcal{E}_2(i\Delta)\hat{x}(i\Delta, 2). \quad (50)$$

Similarly, we have

$$\mathcal{Y}_2(s) = \mathcal{Z}_2(s) - \mathcal{E}_2(s)\hat{x}(s, 2). \quad (51)$$

We shall compute $\mathcal{A}_2(s)$ and $\prod (s)$ as follow. Combining (37) with (39) yields

$$\begin{aligned} e_2(i\Delta + \Delta) &\triangleq x(i\Delta + \Delta) - \hat{x}(i\Delta + \Delta, 2) \\ &= [I + A(i\Delta)\Delta]e_2(i\Delta) \\ &+ \Delta B(i\Delta)x(i\Delta)w(i\Delta) \\ &+ \Delta u(i\Delta) - \mathcal{K}_{p_2}(i\Delta)\mathcal{Y}_2(i\Delta). \end{aligned} \quad (52)$$

Then $\mathcal{A}_2(i\Delta) = \langle e_2(i\Delta), e_2(i\Delta) \rangle$ can be given as

$$\begin{aligned} & \mathcal{A}_2(i\Delta + \Delta) \\ &= \mathcal{A}_2(i\Delta) + \Delta \mathcal{A}_2(i\Delta)A'(i\Delta) \\ &+ \Delta A(i\Delta)\mathcal{A}_2(i\Delta) + A(i\Delta)\mathcal{A}_2(i\Delta)A'(i\Delta)\Delta^2 \\ &+ \Delta B(i\Delta) \prod (i\Delta)B'(i\Delta)M(i\Delta) + \Delta Q(i\Delta) \\ &- \mathcal{K}_{p_2}(i\Delta)R_{w_2}(i\Delta)\mathcal{K}_{p_2}'(i\Delta)/\Delta. \end{aligned} \quad (53)$$

Thus

$$\frac{\mathcal{A}_2(i\Delta + \Delta) - \mathcal{A}_2(i\Delta)}{\Delta}$$

$$\begin{aligned}
&= \mathcal{A}_2(i\Delta)A'(i\Delta) + A(i\Delta)\mathcal{A}_2(i\Delta) \\
&\quad + B(i\Delta)\prod(i\Delta)B'(i\Delta)M(i\Delta) + Q(i\Delta) \quad (54) \\
&\quad - \mathcal{K}_{p_2}(i\Delta)R_{w_2}(i\Delta)\mathcal{K}'_{p_2}(i\Delta)/\Delta^2
\end{aligned}$$

$\Delta \rightarrow 0, i\Delta \rightarrow s$ yields

$$\begin{aligned}
\frac{d\mathcal{A}_2(s)}{ds} &= A(s)\mathcal{A}_2(s) + \mathcal{A}_2(s)A'(s) \\
&\quad + B(s)\prod(s)B'(s)M(s) + Q(s) \quad (55) \\
&\quad - \mathcal{K}_2(s)Q_{w_2}(s)\mathcal{K}'_2(s),
\end{aligned}$$

which is (26).

According to (37), we have

$$\begin{aligned}
\prod(i\Delta + \Delta) &= \prod(i\Delta) + \Delta\prod(i\Delta)A'(i\Delta) + \Delta A(i\Delta)\prod(i\Delta) \\
&\quad + A(i\Delta)\prod(i\Delta)A'(i\Delta)\Delta^2 + \Delta Q(i\Delta) \\
&\quad + \Delta B(i\Delta)\prod(i\Delta)B'(i\Delta)M(i\Delta), \quad (56)
\end{aligned}$$

then

$$\begin{aligned}
\frac{\prod(i\Delta + \Delta) - \prod(i\Delta)}{\Delta} &= \prod(i\Delta)A'(i\Delta) + A(i\Delta)\prod(i\Delta) \\
&\quad + B(i\Delta)\prod(i\Delta)B'(i\Delta)M(i\Delta) \\
&\quad + Q(i\Delta), \quad (57)
\end{aligned}$$

which is (27), this the first part is over.

2) The proof of (30)-(32)

Similar to the proof of part 1), it can be omitted.

3.3. Stochastic optimal filter

Now we are in the position to present the main result in our paper, the following Kalman filtering formulation for the measurement delayed system can be easily given from the proof of Theorem 1.

Theorem 2: Consider the system (1)-(3), optimal estimate $\hat{x}(t|t)$ is given by

$$\hat{x}(t|t) = \hat{x}(t, 1), \quad (58)$$

where $\hat{x}(t, 1)$ is as defined in (13), which is computed in the following steps:

Step 1: Calculate $\hat{x}(s, 2)$ for $s = t_l$:

$$\begin{aligned}
\frac{d\hat{x}(s, 2)}{ds} &= A(s)\hat{x}(s, 2) \\
&\quad + \mathcal{K}_2(s)[\mathcal{Z}_2(s) - \mathcal{E}_2(s)\hat{x}(s, 2)], \quad (59) \\
\hat{x}(0, 2) &= 0,
\end{aligned}$$

where $\mathcal{K}_2(s)$ is as in (28).

Step 2: Compute $\hat{x}(t_l + s, 1)$ for $0 \leq s \leq l$ with the initial value $\hat{x}(t_l, 2)$

$$\frac{d\hat{x}(t_l + s, 1)}{ds} = A(t_l + s)\hat{x}(t_l + s, 1) \quad (60)$$

$$\begin{aligned}
&+ \mathcal{K}_1(t_l + s)[\mathcal{Z}_1(t_l + s) \\
&\quad - \mathcal{E}_1(t_l + s)\hat{x}(t_l + s, 1)],
\end{aligned}$$

$$\hat{x}(t_l, 1) = \hat{x}(t_l, 2),$$

where $\mathcal{K}_1(t_l + s)$ is as in (32).

Step 3: The estimator $\hat{x}(t, 1)$ is computed from Step 2 when $s=l$.

Proof: (58) is from (13), (59) is just (49), and (60) is from Theorem 1.

Remark 7: It can be easily seen that, the new derived stochastic filter is different from the filter from the system without delayed measurement. The new filter involves two parts, for each part, two Riccati equations will be used, but the dimension of the Riccati equations is the same as the origin system.

Remark 8: It should be noted that the proposed approach has given the stochastic optimal filter for continuous-time systems with delayed measurements by the technique of *re-organized innovation analysis*. The advantage of the proposed approach is mainly the computation burden, since it has the same dimension as the origin system. However, the performance of the optimal filter is same as traditional approach.

4. NUMERICAL EXAMPLES

In this section, an example will be given to show the efficiency of the proposed approach. Consider the system (1)-(3) with $l = 0.4s$, and

$$\begin{aligned}
A(t) &= \begin{pmatrix} -4 & -3 \\ 1 & -1 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0.5 & 2 \\ 1 & 1 \end{pmatrix}, \\
C_0(t) &= [0.5t \quad t], \quad C_1(t) = [t \quad 3t], \\
D_0(t) &= [1 \quad 1], \quad D_1(t) = [0.5 \quad 2].
\end{aligned}$$

The initial state value $x(0) = [0.5 \quad 1]$, $\hat{x}(0) = [0 \quad 0]'$ and $u(t), w(t), v_0(t)$, and $v_1(t)$ are uncorrelated white noises with zero means and unity covariance matrices, i.e.,

$$P_0(t) = I_2, \quad Q(t) = I_2, \quad M(t) = 1, \quad R_0(t) = R_1(t) = 1.$$

Our aim is to calculate the optimal estimate $\hat{x}(t|t)$ of the signal $x(t)$ based on observation $\{y_0(\tau)|_{0 \leq \tau \leq t}, y_1(\tau)|_{l \leq \tau \leq t}\}$. In this example, we assume that $t \geq 0.4s$. When $0 \leq \tau \leq 0.4s$, the problem becomes a standard Kalman filtering for the system without the delayed measurement. To compute the filter, we re-organize the linear space $\mathcal{L}\{\{y_0(\tau)|_{0 \leq \tau \leq t}, y_1(\tau)|_{l \leq \tau \leq t}\}\}$ equivalently as the following linear space

$$\mathcal{L}\{\mathcal{Z}_2(\tau)|_{0 \leq \tau \leq t_l}; \mathcal{Z}_1(\tau)|_{t_l \leq \tau \leq t}\}, \quad (61)$$

where $t_1 = t - 0.4$, and the new observation $\mathcal{Z}(\tau)$ ($i=1,2$) are

$$\begin{aligned} \mathcal{Z}_1(\tau) &= [y_0^T(\tau) \quad y_1^T(\tau + 0.4)]^T, \\ \mathcal{Z}_2(\tau) &= y_0(\tau), \end{aligned} \quad (62)$$

satisfying

$$\mathcal{X}_1(\tau) = \mathcal{G}_1(\tau)x(\tau) + \mathcal{D}_1(\tau)x(\tau)w(\tau) + \mathcal{Z}_1(\tau), \quad (63)$$

$$\mathcal{X}_2(\tau) = \mathcal{G}_2(\tau)x(\tau) + \mathcal{D}_2(\tau)x(\tau) + \mathcal{Z}_2(\tau), \quad (64)$$

with

$$\begin{aligned} \mathcal{G}_2(\tau) &= \begin{pmatrix} 0.5\tau & \tau \\ \tau & 3\tau \end{pmatrix}, \quad \mathcal{G}_1(\tau) = [0.5\tau \quad \tau], \\ \mathcal{D}_2(\tau) &= \begin{pmatrix} w(\tau) & w(\tau) \\ 0.5w(\tau + 0.4) & 2w(\tau + 0.4) \end{pmatrix}, \\ \mathcal{A}(\tau) &= [1 \quad 1], \quad \mathcal{Z}_2(\tau) = \begin{bmatrix} v_0(\tau) \\ v_1(\tau + 0.4) \end{bmatrix}, \\ \mathcal{Z}_1(\tau) &= v_0(\tau), \end{aligned} \quad (65)$$

and $\mathcal{Z}_i(t)$ ($i=1,2$) are white noises with zero means and unity covariance $Q_{\mathcal{Z}_i}(t) = I_i$.

Having obtained the re-organized observation sequence, the computation of the optimal filter $\hat{x}(t|t)$ can be summarized by three steps below:

Step 1: Compute \mathcal{Z}^T by (26) with $l = 0.4s$ and initial value $\mathcal{Z}^3 = 0$. Compute $\hat{x}(\tau, 2)$ by (60) with initial value $\hat{x}(0, 2) = 0$.

Step 2: Note $\mathcal{Z}_1^{t_1} \equiv \mathcal{Z}_2^{t_1}$ and $\hat{x}(t_1, 1) \equiv \hat{x}(t_1, 2)$. $\mathcal{Z}_1^{t_1+\tau}$ and $\hat{x}(t_1 + \tau, 1)$ are calculated by (30) and (61), respectively, with $i=1$ and initial values $\mathcal{Z}_1^{t_1} \equiv \mathcal{Z}_2^{t_1}$ and $\hat{x}(t_1, 1) \equiv \hat{x}(t_1, 2)$.

Step 3: Calculate the optimal filter $\hat{x}(t|t)$ in (58).

In the simulation, the sampling period is $T = 0.02s$ Fig. 1. The estimate for $x_1(t)$ with delayed measurement and time horizon is $T = 4s$.

We denote $\hat{x}(t|t) = [\hat{x}_1(t|t), \hat{x}_2(t|t)]^T$. The tracking performance of the filter $\hat{x}(t|t)$ with the time delayed measurement is given in Fig. 1 whereas $\hat{x}(t|t)$ is given in Fig. 2. In Fig. 3, the comparison of the sum of error variance between the cases for the system with delay-free measurement and time-delayed measurement is given. In Figs. 1 and 2, the solid line denotes the true signal and the dashed line is its estimate. In Fig. 3, the solid line denotes the sum of error variance for the case when only the delay free measurement is used and the dashed line is the sum of error variance for the case when the delay free

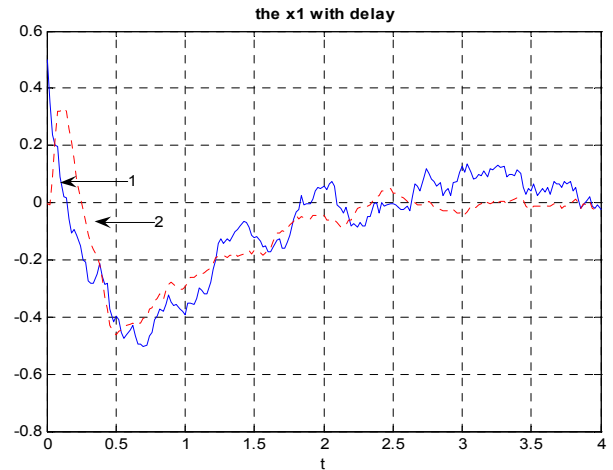


Fig. 1. The estimate for $x_1(t)$ with delayed, measurement where 1 is the original state and 2 is the estimate.

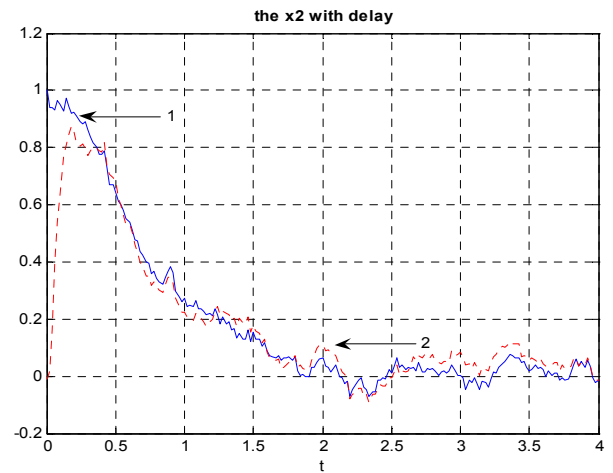


Fig. 2. The estimate for $x_2(t)$ with delayed measurement where 1 is the original state and 2 is the estimate.

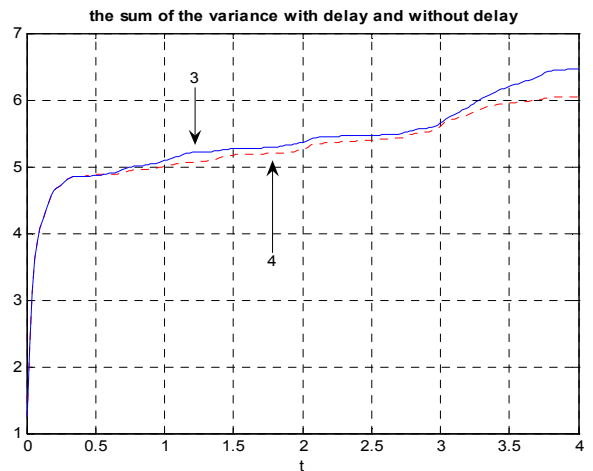


Fig. 3. The comparison for the sum of error covariance where 3 is for without delay and 4 is for with delayed measurement.

measurement and the time-delayed measurement is used. It can be observed that with additional delayed measurement, the filtering performance is significantly improved as expected.

5. CONCLUSION

The paper has studied the stochastic Kalman filtering problem for a linear continuous-time system with both instantaneous and time-delayed measurements. Multiplicative white noise is introduced in the state-space model. By using *re-organized innovation analysis*, the new stochastic filter is presented. The solution to the derived Kalman filter is given in terms of Riccati differential equations. More importantly, the proposed approach in the paper can be applied to solve many complicated problems such as stochastic H_∞ estimation [20], H_∞ control stochastic system with preview and so on.

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