Robust $\mathcal{D}$-Stability and $\mathcal{D}$-Stabilization of Dynamic Interval Systems

Wei-Jie Mao and Jian Chu

Abstract: A sufficient condition for the robust $\mathcal{D}$-stability of dynamic interval systems is proposed in this paper. This $\mathcal{D}$-stability condition is based on a parameter-dependent Lyapunov function obtained from the feasibility of a set of matrix inequalities defined at a series of partial-vertex-based interval matrices other than the total vertex matrices as previous results. This condition is also extended to the robust $\mathcal{D}$-stabilization problem of dynamic interval systems, which supplies an effective synthesis procedure for any LMI $\mathcal{D}$-region. The proposed conditions can be simplified to a set of LMIs, which can be solved by efficient interior point methods in polynomial time.

Keywords: Interval systems, parameter-dependent Lyapunov function, robust $\mathcal{D}$-stability, robust $\mathcal{D}$-stabilization.

1. INTRODUCTION

Robust stability problem of interval matrices has attracted considerable attention over last decades. Lots of results are readily available, e.g., [1-7]. Most of existing results constitute sufficient conditions for robust stability of interval matrices. Among those necessary and sufficient conditions, two main streams may be quoted. The first one is based on the finite subdivisions of interval matrices (see e.g., [3,4]). The computation amount depends on the conservatism of applied sufficient conditions over each of subinterval matrices. The second one is in terms of the total vertices or exposed faces (see e.g., [5-7]). This number of vertices or faces, except for the systems of low dimension, is obviously enormous and it results in heavy computation. More recently, based on parameter-dependent Lyapunov approach, robust $\mathcal{D}$-stability analysis for uncertain polytopic systems appeared in [8,9]. The results encompass the usual stability of continuous-time and discrete-time uncertain systems as particular cases. As an extension, a linearizing change of variables was proposed in [8] to get sufficient conditions of $\mathcal{D}$-stabilization of a polytope of matrices. However, this change of variables is no more valid for some $\mathcal{D}$-regions such as half-planes. In [10], a sufficient condition of robust $\mathcal{D}$-stabilization of a polytope of matrices was characterized by an LMI involving matrix variables subject to an additional non-linear algebraic constraint. As the conic complementarity formulation and related numerical procedure used, this approach still suffers from its computational complexity.

In this paper, a new sufficient condition for the quadratic $\mathcal{D}$-stability of dynamic interval systems is firstly derived. To further reduce the conservatism of quadratic $\mathcal{D}$-stability, a parameter-dependent Lyapunov function is introduced into the analysis of dynamic interval systems. A partial-vertex-based condition other than a total-vertex-based condition is proposed for the robust $\mathcal{D}$-stability of dynamic interval systems. Compared with the total-vertex-based approach, the computation amount of the proposed partial-vertex-based method can be reduced in the most cases. The results are also extended to the $\mathcal{D}$-stabilization of dynamic interval systems, which supplies an effective synthesis procedure for any LMI $\mathcal{D}$-region. Moreover, these conditions can be simplified to a set of LMIs and the LMI feasibility problem can be solved by efficient interior point methods in polynomial time. Thus, the proposed approach in this paper leads to a less computational complexity than that in [10].

2. PRELIMINARIES

The following notations will be used throughout the paper. R^n denotes the n dimensional Euclidean space and $\mathrm{R}^{n\times m}$ is the set of all $n\times m$ real matrices. The notation $X \geq Y$ (respectively, $X > Y$), where X and Y are symmetric matrices, means that the matrix $X - Y$ is positive semi-definite (respectively, positive

Manuscript received May 31, 2005; revised September 5, 2006; accepted January 30, 2007. Recommended by Editorial Board member Lihua Xie under the direction of Editor Tae-Woong Yoon. This work was supported by the 973 Program (No. 2006CB705400) and the National Natural Science Foundation of China (No. 60421002).

Wei-Jie Mao and Jian Chu are with the State Key Lab of Industrial Control Technology, Institute of Advanced Process Control, Zhejiang University, Hangzhou 310027, China (e-mails: {wjmao, chuj}@iipc.zju.edu.cn).

definite). $\otimes$ is the Kronecker product of two matrices. The symbol $*$ will be used to induce a symmetric structure. For example, $\begin{bmatrix} X & * \\ Z & Y \end{bmatrix} = \begin{bmatrix} X & Z^T \\ Z & Y \end{bmatrix}$. For all $1 \le i, j \le n$, with $A^m = [a_{ij}^m]_{n\times n}$, $A^M = [a_{ij}^M]_{n\times n}$ satisfying $a_{ij}^m \le a_{ij}^M$, we define an interval matrix as

$$\mathcal{A} = \left\{[a_{ij}]_{n\times n} : a_{ij}^m \le a_{ij} \le a_{ij}^M, 1 \le i, j \le n\right\}. \tag{1}$$

Consider the following dynamic interval system

$$\delta[x(t)] = Ax(t), \tag{2}$$

where $x(t) \in \mathrm{R}^n$ is the state, $\delta[\cdot]$ denotes the time derivative for continuous-time systems and shift operator for discrete-time systems, and $A \in \mathcal{A}$. Let

$$A_0 = \frac{1}{2}(A^m + A^M), \tag{3}$$

$$\Delta A = \frac{1}{2}(A^M - A^m). \tag{4}$$

Then A can be written as

$$A = A_0 + \sum_{i,j=1}^{n} e_i f_{ij} e_j^T, \left| f_{ij} \right| \le \Delta a_{ij}, \tag{5}$$

where $e_k \in \mathrm{R}^n$ denotes the column vector with k th element to be 1 and others to be 0.

Consider the region of the complex plain defined by

$$\mathcal{D} = \left\{z \in \mathrm{C} : R_{11} + R_{12}z + R_{12}^T z^* + R_{22}zz^* < 0\right\}, \tag{6}$$

where $R_{11} = R_{11}^T \in \mathrm{R}^{d\times d}$ and $R_{22} = R_{22}^T \in \mathrm{R}^{d\times d}$ are submatrices of $R \in \mathrm{R}^{2d\times 2d}$ such that

$$R = \left[\begin{array}{c|c} R_{11} & R_{12} \\ \hline R_{12}^T & R_{22} \end{array}\right], \tag{7}$$

and d is called the order of the $\mathcal{D}$-region. It is assumed that $R_{22} \ge 0$ and, therefore, $\mathcal{D}$ defined in (6) represents convex regions symmetric with respect to the real axis.

Remark 1: Typical $\mathcal{D}$-regions used in the usual stability analysis are the left-hand side of complex plane (continuous-time systems) and the unitary disk centered at the origin (discrete-time systems) represented, respectively, by the choices of R

$$R_C = \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right], \quad R_D = \left[\begin{array}{c|c} -1 & 0 \\ \hline 0 & 1 \end{array}\right]. \tag{8}$$

Definition 1 [8]**:** A matrix $A \in \mathrm{R}^{n\times n}$ is said to be $\mathcal{D}$-stable if all its eigenvalues lie in the $\mathcal{D}$-region specified by (6).

Lemma 1 [8]**:** A matrix $A \in \mathrm{R}^{n\times n}$ is *$\mathcal{D}$-stable* if and only if there exists a symmetric positive definite matrix $P \in \mathrm{R}^{n\times n}$ such that

$$\begin{aligned} &R_{11} \otimes P + R_{12} \otimes (PA) \\ &\quad + R_{12}^T \otimes (A^T P) + R_{22} \otimes (A^T PA) < 0. \end{aligned} \tag{9}$$

3. ROBUST $\mathcal{D}$-STABILITY CONDITIONS

Definition 2 [8]**:** $\mathcal{A}$ is said to be *quadratically $\mathcal{D}$-stable* if there exists a symmetric positive definite matrix $P \in \mathrm{R}^{n\times n}$ such that

$$\begin{aligned} &R_{11} \otimes P + R_{12} \otimes (PA) \\ &\quad + R_{12}^T \otimes (A^T P) + R_{22} \otimes (A^T PA) < 0 \end{aligned} \tag{10}$$

for all $A \in \mathcal{A}$.

Definition 3 [8]**:** $\mathcal{A}$ is said to be *robustly $\mathcal{D}$-stable* if, for all $A \in \mathcal{A}$, A is $\mathcal{D}$-stable.

Theorem 1: $\mathcal{A}$ is *quadratically $\mathcal{D}$-stable* if there exist a symmetric positive definite matrix $P \in \mathrm{R}^{n\times n}$, a matrix $G \in \mathrm{R}^{n\times n}$ and real scalars ε, $\lambda_{ij} > 0$ $(i, j = 1, 2, \cdots, n)$ such that

$$\begin{bmatrix} \Phi & * & * \\ \begin{pmatrix} \varepsilon^{-1} R_{12}^T \otimes (P - G) \\ +\varepsilon I_d \otimes (A_0 G)^T \end{pmatrix} & \begin{pmatrix} -I_d \otimes (G + G^T) \\ +\varepsilon^{-2} R_{22} \otimes P \end{pmatrix} & * \\ U^T (R_{12}^T \otimes G) & \varepsilon U^T \left(I_d \otimes G\right) & -V \end{bmatrix} < 0 \tag{11}$$

where

$$\begin{aligned} \Phi = &R_{11} \otimes P + R_{12} \otimes (A_0 G)^T + R_{12}^T \otimes (A_0 G) \\ &+ \sum_{i,j=1}^{n} \lambda_{ij} \Delta a_{ij}^2 \left(I_d \otimes (e_i e_i^T)\right), \end{aligned} \tag{12}$$

$$\begin{aligned} U = &\left[I_d \otimes e_1 \ \cdots \ I_d \otimes e_n \right. \\ &\left. \cdots \ I_d \otimes e_1 \ \cdots \ I_d \otimes e_n\right], \end{aligned} \tag{13}$$

$$\begin{aligned} V = \mathrm{diag}&\left\{I_d \otimes \lambda_{11} \ \cdots \ I_d \otimes \lambda_{1n} \right. \\ &\left. \cdots \ I_d \otimes \lambda_{n1} \ \cdots \ I_d \otimes \lambda_{nn}\right\}. \end{aligned} \tag{14}$$

Proof: Define

$$Y = \begin{bmatrix} \begin{pmatrix} R_{11} \otimes P + R_{12} \otimes (AG)^T \\ +R_{12}^T \otimes (AG) \end{pmatrix} & * \\ \begin{pmatrix} \varepsilon^{-1} R_{12}^T \otimes (P - G) \\ +\varepsilon I_d \otimes (AG)^T \end{pmatrix} & \begin{pmatrix} -I_d \otimes (G + G^T) \\ +\varepsilon^{-2} R_{22} \otimes P \end{pmatrix} \end{bmatrix}$$

with a symmetric positive definite matrix $P \in \mathrm{R}^{n\times n}$, a matrix $G \in \mathrm{R}^{n\times n}$ and a positive real scalar ε. Then, for any real scalars $\lambda_{ij} > 0$ $(i, j = 1, 2, \cdots, n)$,

$$Y = \begin{bmatrix} \begin{pmatrix} R_{11} \otimes P + R_{12}^T \otimes (A_0 G) \\ + R_{12} \otimes (A_0 G)^T \end{pmatrix} & * \\ \begin{pmatrix} \varepsilon^{-1} R_{12}^T \otimes (P - G) \\ + \varepsilon I_d \otimes (A_0 G)^T \end{pmatrix} & \begin{pmatrix} -I_d \otimes (G + G^T) \\ + \varepsilon^{-2} R_{22} \otimes P \end{pmatrix} \end{bmatrix}$$
$$+ \sum_{i,j=1}^{n} \left\{ \begin{bmatrix} R_{12} \otimes G^T \\ \varepsilon I_d \otimes G^T \end{bmatrix} \left(I_d \otimes (e_j f_{ij} e_i^T) \right) \begin{bmatrix} I_{dn} & 0 \end{bmatrix} + \begin{bmatrix} I_{dn} \\ 0 \end{bmatrix} \left(I_d \otimes (e_i f_{ij} e_j^T) \right) \begin{bmatrix} R_{12}^T \otimes G & \varepsilon I_d \otimes G \end{bmatrix} \right\}$$
$$\leq \begin{bmatrix} \begin{pmatrix} R_{11} \otimes P + R_{12}^T \otimes (A_0 G) \\ + R_{12} \otimes (A_0 G)^T \end{pmatrix} & * \\ \begin{pmatrix} \varepsilon^{-1} R_{12}^T \otimes (P - G) \\ + \varepsilon I_d \otimes (A_0 G)^T \end{pmatrix} & \begin{pmatrix} -I_d \otimes (G + G^T) \\ + \varepsilon^{-2} R_{22} \otimes P \end{pmatrix} \end{bmatrix}$$
$$+ \sum_{i,j=1}^{n} \left\{ \frac{1}{\lambda_{ij}} \begin{bmatrix} R_{12} \otimes G^T \\ \varepsilon I_d \otimes G^T \end{bmatrix} \left(I_d \otimes (e_j e_j^T) \right) \begin{bmatrix} R_{12}^T \otimes G & \varepsilon I_d \otimes G \end{bmatrix} + \lambda_{ij} \Delta a_{ij}^2 \begin{bmatrix} I_{dn} \\ 0 \end{bmatrix} \left(I_d \otimes (e_i e_i^T) \right) \begin{bmatrix} I_{dn} & 0 \end{bmatrix} \right\} < 0.$$

The right part of the above inequality is derived from the Schur complement of (11). Furthermore, noting that

$$\begin{bmatrix} I_{dn} \\ \varepsilon I_d \otimes A^T \end{bmatrix}^T Y \begin{bmatrix} I_{dn} \\ \varepsilon I_d \otimes A^T \end{bmatrix} = R_{11} \otimes P + R_{12} \otimes (PA^T) + R_{12}^T \otimes (AP) + R_{22} \otimes (APA^T) < 0,$$

which is exactly the inequality (10) with A replaced by A^T, it follows immediately from Definition 2 that $\mathcal{A}$ is quadratically $\mathcal{D}$-stable. □

Remark 2: The condition (11) is actually an LMI when the real scalar ε in Theorem 1 is a priori given. The variable ε in (11) supplies an additional degree of freedom for the feasibility of (11). ε can be initially set as 1. If LMI (11) is infeasible, a possible solution may be searched by tuning ε or by iterating over ε.

Some notations are necessary to state the main result concerning the robust $\mathcal{D}$-stability of an interval matrix $\mathcal{A}$. Let $J = \{(i, j) : i, j = 1, 2, \cdots, n\}$. Then, for $J_1 \subseteq J$, a series of partial-vertex-based interval matrices are defined as

$$\mathcal{A}_{(k)} = \Big\{ [a_{(k)ij}]_{n\times n} : a_{ij}^m \leq a_{(k)ij} \leq a_{ij}^M \text{ for } (i, j) \notin J_1; \quad a_{(k)ij} = a_{ij}^m \text{ or } a_{(k)ij} = a_{ij}^M \text{ for } (i, j) \in J_1 \Big\},$$
$$k = 1, 2, \cdots, 2^N, \tag{15}$$

where N is the number of elements in J_1. $A_{(k)0}$, $\Delta A_{(k)}$ can be defined similarly as (3) and (4).

Theorem 2: Given $J_1 \subseteq J$, $\mathcal{A}$ is *robustly $\mathcal{D}$-stable* if there exist symmetric positive definite matrices $P_{(k)} \in \mathrm{R}^{n\times n}$, a matrix $G \in \mathrm{R}^{n\times n}$ and real scalars ε, $\lambda_{(k)ij} > 0$ $(i, j = 1, 2, \cdots, n)$ such that

$$\begin{bmatrix} \Phi_{(k)} & * & * \\ \begin{pmatrix} \varepsilon^{-1} R_{12}^T \otimes (P_{(k)} - G) \\ + \varepsilon I_d \otimes (A_{(k)0} G)^T \end{pmatrix} & \begin{pmatrix} -I_d \otimes (G + G^T) \\ + \varepsilon^{-2} R_{22} \otimes P_{(k)} \end{pmatrix} & * \\ U^T (R_{12}^T \otimes G) & \varepsilon U^T \left(I_d \otimes G \right) & -V_{(k)} \end{bmatrix} < 0,$$
$$k = 1, 2, \cdots, 2^N, \tag{16}$$

where U is defined as (13) and $\Phi_{(k)}, V_{(k)}$ are defined as

$$\Phi_{(k)} = R_{12} \otimes (A_{(k)0} G)^T + R_{12}^T \otimes (A_{(k)0} G) + R_{11} \otimes P_{(k)} + \sum_{i,j=1}^{n} \lambda_{(k)ij} \Delta a_{(k)ij}^2 \left(I_d \otimes (e_i e_i^T) \right), \tag{17}$$

$$V_{(k)} = \mathrm{diag}\Big\{ I_d \otimes \lambda_{(k)11} \quad \cdots \quad I_d \otimes \lambda_{(k)1n} \quad \cdots \quad I_d \otimes \lambda_{(k)n1} \quad \cdots \quad I_d \otimes \lambda_{(k)nn} \Big\}. \tag{18}$$

Proof: By Theorem 1, $\mathcal{A}_{(k)}$ is quadratically $\mathcal{D}$-stable from (16). It means that every vertex matrix A_i of $\mathcal{A}_{(k)}$, which is also the vertex matrix of $\mathcal{A}$, satisfying

$$\begin{bmatrix} \begin{pmatrix} R_{11} \otimes P_i + R_{12}^T \otimes (A_i G) \\ + R_{12} \otimes (A_i G)^T \end{pmatrix} & * \\ \begin{pmatrix} \varepsilon^{-1} R_{12}^T \otimes (P_i - G) \\ + \varepsilon I_d \otimes (A_i G)^T \end{pmatrix} & \begin{pmatrix} -I_d \otimes (G + G^T) \\ + \varepsilon^{-2} R_{22} \otimes P_i \end{pmatrix} \end{bmatrix} < 0,$$
$$i = 1, 2, \cdots, 2^{n\times n},$$

where P_i corresponds to one of $P_{(k)}$, $k = 1, 2, \cdots, 2^N$. Multiply the above $i = 1, 2, \cdots, 2^{n\times n}$ inequalities by

τ_i, $\tau_i \geq 0$, $\sum_{i=1}^{2^{n\times n}} \tau_i = 1$, and sum to get

$$Y = \begin{bmatrix} \Xi(\tau) & * \\ \begin{pmatrix} \varepsilon^{-1} R_{12}^T \otimes (P(\tau) - G) \\ +\varepsilon I_d \otimes (A(\tau)G)^T \end{pmatrix} & \begin{pmatrix} -I_d \otimes (G + G^T) \\ +\varepsilon^{-2} R_{22} \otimes P(\tau) \end{pmatrix} \end{bmatrix} < 0,$$

where

$$\Xi(\tau) = R_{11} \otimes P(\tau) + R_{12}^T \otimes (A(\tau)G) + R_{12} \otimes (A(\tau)G)^T,$$

$$P(\tau) = \sum_{i=1}^{2^{n\times n}} \tau_i P_i, \quad A(\tau) = \sum_{i=1}^{2^{n\times n}} \tau_i A_i.$$

Following the same procedure as in the proof of Theorem 1, we get

$$R_{11} \otimes P(\tau) + R_{12} \otimes \left(P(\tau)A^T(\tau)\right) + R_{12}^T \otimes (A(\tau)P(\tau)) + R_{22} \otimes \left(A(\tau)P(\tau)A^T(\tau)\right) < 0.$$

Then, it follows immediately from Lemma 1 and Definition 3 that $\mathcal{A}$ is robustly $\mathcal{D}$-stable. □

Remark 3: There are 2^N inequalities in Theorem 2. In the worst case of $J_1 = J$, this number increases to $2^{n\times n}$ and Theorems 2 recover the total-vertex-based conditions. However, in the most cases, N is less than n^2. Thus, compared with those total-vertex-based conditions, the computation amount of proposed partial-vertex-based conditions can be reduced in the most cases.

4. ROBUST $\mathcal{D}$-STABILIZATION CONDITIONS

Let us consider the following dynamic interval system

$$\delta[x(t)] = Ax(t) + Bu(t), \tag{19}$$

where $x(t) \in \mathrm{R}^n$ is the state; $u(t) \in \mathrm{R}^p$ is the control input; $A \in \mathcal{A}$ and $B \in \mathcal{B}$ with $\mathcal{A}$ defined as (1) and $\mathcal{B}$ defined similarly as

$$\mathcal{B} = \left\{[b_{ij}]_{n\times p} : b_{ij}^m \leq b_{ij} \leq b_{ij}^M, 1 \leq i \leq n, 1 \leq j \leq p\right\}, \tag{20}$$

where $B^m = [b_{ij}^m]_{n\times p}$ and $B^M = [b_{ij}^M]_{n\times p}$ satisfy $b_{ij}^m \leq b_{ij}^M$ for all $1 \leq i \leq n, 1 \leq j \leq p$. Let

$$B_0 = \frac{1}{2}(B^m + B^M), \tag{21}$$

$$\Delta B = \frac{1}{2}(B^M - B^m). \tag{22}$$

Then B can be written as

$$B = B_0 + \sum_{i=1}^{n}\sum_{j=1}^{p} e_i g_{ij} h_j^T, \left|g_{ij}\right| \leq \Delta b_{ij}, \tag{23}$$

where $e_k \in \mathrm{R}^n$ or $h_k \in \mathrm{R}^p$ denotes the column vector with k th element to be 1 and others to be 0.

Definition 4: The dynamic interval system (19) is said to be *quadratically $\mathcal{D}$-stabilizable* if there exist a symmetric positive definite matrix $P \in \mathrm{R}^{n\times n}$ and a matrix $K \in \mathrm{R}^{p\times n}$ such that

$$\begin{aligned} &R_{11} \otimes P + R_{12} \otimes [P(A + BK)] \\ &\quad + R_{12}^T \otimes \left[(A + BK)^T P\right] \\ &\quad + R_{22} \otimes \left[(A + BK)^T P(A + BK)\right] < 0 \end{aligned} \tag{24}$$

for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. K is called as a quadratically $\mathcal{D}$-stabilizing matrix.

Definition 5: The dynamic interval system (19) is said to be *robustly $\mathcal{D}$-stabilizable* if, for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, there exists a matrix $K \in \mathrm{R}^{p\times n}$ such that $A + BK$ is $\mathcal{D}$-stable. K is called as a robustly $\mathcal{D}$-stabilizing matrix.

Theorem 3: The dynamic interval system (19) is *quadratically $\mathcal{D}$-stabilizable* if there exist a symmetric positive definite matrix $P \in \mathrm{R}^{n\times n}$, matrices $G \in \mathrm{R}^{n\times n}$ and $Z \in \mathrm{R}^{p\times n}$, and real scalars ε, $\lambda_{ij} > 0$ $(i, j = 1, 2, \cdots, n)$, $\delta_{ij} > 0$ $(i = 1, 2, \cdots, n; j = 1, 2, \cdots, p)$ such that

$$\begin{bmatrix} \Psi & * & * & * \\ \begin{pmatrix} \varepsilon I_d \otimes (A_0 G + B_0 Z)^T \\ +\varepsilon^{-1} R_{12}^T \otimes (P - G) \end{pmatrix} & \begin{pmatrix} -I_d \otimes (G + G^T) \\ +\varepsilon^{-2} R_{22} \otimes P \end{pmatrix} & * & * \\ U_1^T (R_{12}^T \otimes G) & \varepsilon U_1^T (I_d \otimes G) & -V_1 & * \\ U_2^T (R_{12}^T \otimes Z) & \varepsilon U_2^T (I_d \otimes Z) & 0 & -V_2 \end{bmatrix} < 0, \tag{25}$$

where

$$\begin{aligned} \Psi &= R_{12} \otimes (A_0 G + B_0 Z)^T + R_{12}^T \otimes (A_0 G + B_0 Z) \\ &\quad + R_{11} \otimes P + \sum_{i,j=1}^{n} \lambda_{ij} \Delta a_{ij}^2 \left(I_d \otimes (e_i e_i^T)\right) \\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{p} \delta_{ij} \Delta b_{ij}^2 \left(I_d \otimes (e_i e_i^T)\right), \end{aligned} \tag{26}$$

$$\begin{aligned} U_1 = [&I_d \otimes e_1 \ \cdots \ I_d \otimes e_n \\ &\cdots \ I_d \otimes e_1 \ \cdots \ I_d \otimes e_n], \end{aligned} \tag{27}$$

$$\begin{aligned} U_2 = [&I_d \otimes e_1 \ \cdots \ I_d \otimes e_n \\ &\cdots \ I_d \otimes e_1 \ \cdots \ I_d \otimes e_n], \end{aligned} \tag{28}$$

$$\begin{aligned} V_1 = \mathrm{diag}\{&I_d \otimes \lambda_{11} \ \cdots \ I_d \otimes \lambda_{1n} \\ &\cdots \ I_d \otimes \lambda_{n1} \ \cdots \ I_d \otimes \lambda_{nn}\}, \end{aligned} \tag{29}$$

$$V_2 = \operatorname{diag}\left\{ I_d \otimes \delta_{11} \quad \cdots \quad I_d \otimes \delta_{1p} \quad \cdots \quad I_d \otimes \delta_{n1} \quad \cdots \quad I_d \otimes \delta_{np} \right\}. \tag{30}$$

Furthermore, the quadratically $\mathcal{D}$-stabilizing state feedback matrix is $K = ZG^{-1}$.

Proof: The proof is similar to that of Theorem 1, based on Definition 4, by replacing A with $A + BK$ and setting $Z = KG$. □

Let $C^m = [A^m, B^m]$, $C^M = [A^M, B^M]$, and $J = \{(i, j) : i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, n + p\}$. Then, for $J_1 \subseteq J$, a series of partial-vertex-based interval matrices are defined as

$$\mathcal{C}_{(k)} = [\mathcal{A}_{(k)}, \mathcal{B}_{(k)}] = \left\{ \begin{array}{l} [c_{(k)ij}]_{n\times(n+p)} : \\ c_{ij}^m \le c_{(k)ij} \le c_{ij}^M \ \text{ for } (i,j) \notin J_1; \\ c_{(k)ij} = c_{ij}^m \text{ or } c_{(k)ij} = c_{ij}^M \ \text{ for } (i,j) \in J_1 \end{array} \right\}, \tag{31}$$
$$k = 1, 2, \cdots, 2^N,$$

where N is the number of elements in J_1. $C_{(k)0} = [A_{(k)0}, B_{(k)0}]$ and $\Delta C_{(k)} = [\Delta A_{(k)}, \Delta B_{(k)}]$ can be defined similarly as (3)-(4) and (21)-(22).

Theorem 4: Given $J_1 \subseteq J$, the dynamic interval system (19) is *robustly $\mathcal{D}$-stabilizable* if there exist symmetric positive definite matrices $P_{(k)} \in \mathrm{R}^{n\times n}$, matrices $G \in \mathrm{R}^{n\times n}$ and $Z \in \mathrm{R}^{p\times n}$, and real scalars ε, $\lambda_{(k)ij} > 0$ $(i, j = 1, 2, \cdots, n)$, $\delta_{(k)ij} > 0$ $(i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, p)$ such that

$$\begin{bmatrix} \Psi_{(k)} & * & * & * \\ \begin{pmatrix} \varepsilon^{-1} R_{12}^T \otimes (P_{(k)} - G) \\ +\varepsilon I_d \otimes (A_{(k)0} G \\ +B_{(k)0} Z)^T \end{pmatrix} & \begin{pmatrix} -I_d \otimes (G + G^T) \\ +\varepsilon^{-2} R_{22} \otimes P_{(k)} \end{pmatrix} & * & * \\ U_1^T (R_{12}^T \otimes G) & \varepsilon U_1^T \left(I_d \otimes G \right) & -V_{(k)1} & * \\ U_2^T (R_{12}^T \otimes Z) & \varepsilon U_2^T \left(I_d \otimes Z \right) & 0 & -V_{(k)2} \end{bmatrix} < 0,$$
$$k = 1, 2, \cdots, 2^N, \tag{32}$$

where U_1, U_2 are defined as (27)-(28), and $\Psi_{(k)}$, $V_{(k)1}$, $V_{(k)2}$ are defined as

$$\begin{aligned} \Psi_{(k)} = {} & R_{11} \otimes P_{(k)} + R_{12} \otimes (A_{(k)0} G + B_{(k)0} Z)^T \\ & + R_{12}^T \otimes (A_{(k)0} G + B_{(k)0} Z) \\ & + \sum_{i,j=1}^{n} \lambda_{(k)ij} \Delta a_{(k)ij}^2 \left(I_d \otimes (e_i e_i^T) \right) \\ & + \sum_{i=1}^{n} \sum_{j=1}^{p} \delta_{(k)ij} \Delta b_{(k)ij}^2 \left(I_d \otimes (e_i e_i^T) \right), \end{aligned} \tag{33}$$

$$V_{(k)1} = \operatorname{diag}\left\{ I_d \otimes \lambda_{(k)11} \quad \cdots \quad I_d \otimes \lambda_{(k)1n} \quad \cdots \quad I_d \otimes \lambda_{(k)n1} \quad \cdots \quad I_d \otimes \lambda_{(k)nn} \right\}, \tag{34}$$

$$V_{(k)2} = \operatorname{diag}\left\{ I_d \otimes \delta_{(k)11} \quad \cdots \quad I_d \otimes \delta_{(k)1p} \quad \cdots \quad I_d \otimes \delta_{(k)n1} \quad \cdots \quad I_d \otimes \delta_{(k)np} \right\}. \tag{35}$$

Furthermore, the robustly $\mathcal{D}$-stabilizing state feedback matrix is $K = ZG^{-1}$.

Proof: The proof is similar to that of Theorem 2, based on Definition 5, by replacing A with $A + BK$ and setting $Z = KG$. □

Remark 4: The proposed conditions for $\mathcal{D}$-stabilization are applicable to the usual stabilization of both continuous-time and discrete-time systems. Thus, they are better than the result in [8], which is only applicable to the case of discrete-time systems.

5. NUMERICAL EXAMPLES

Example 1: Consider the $\mathcal{D}$-stabilization problem of the interval continuous-time system defined by

$$A^m = \begin{bmatrix} -7 & 4.6 & 4.5 & -5.3 \\ -4.3 & -6.1 & -3.5 & 0.2 \\ -5.7 & 2.6 & -7.6 & -0.2 \\ -2.8 & 0.5 & 4.3 & -4.6 \end{bmatrix}, \quad B^m = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix},$$

$$A^M = \begin{bmatrix} -3 & 7.1 & 6.3 & -2.7 \\ -1.7 & -2.8 & -1.5 & 2 \\ -4 & 4.9 & -3.1 & 3.2 \\ -0.6 & 2.5 & 6.2 & -2.8 \end{bmatrix}, \quad B^M = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

This system is obviously unstable. By Theorem 3 $(R = R_C)$, LMI (25) is feasible with $\varepsilon = 0.1$ and this system is determined to be quadratically $\mathcal{D}$-stabilizable. Furthermore, let us add a constraint $\mathrm{Re}(z) < -0.89$ on the $\mathcal{D}$-region, i.e., $R = \left[\begin{array}{c|c} 1.78 & 1 \\ \hline 1 & 0 \end{array} \right]$, then the above system cannot be determined to be quadratically $\mathcal{D}$-stabilizable by Theorem 3. But, by Theorem 4, LMI (32) is feasible with $\varepsilon = 0.1$ and this system can be determined to be robustly $\mathcal{D}$-stabilizable. The partial-vertex-based interval matrices $\mathcal{C}_{(k)}$ (k=1,2) are defined as

$$C_{(1)}^{m}=\begin{bmatrix}-7 & 4.6 & 4.5 & -5.3 & 2\\ -4.3 & -6.1 & -3.5 & 0.2 & 0\\ -5.7 & 2.6 & -7.6 & -0.2 & -2\\ -2.8 & 0.5 & 4.3 & -4.6 & 0\end{bmatrix},$$

$$C_{(1)}^{M}=\begin{bmatrix}-3 & 7.1 & 6.3 & -2.7 & 3\\ -1.7 & -2.8 & -1.5 & 2 & 0\\ -4 & 4.9 & -7.6 & 3.2 & -1\\ -0.6 & 2.5 & 6.2 & -2.8 & 0\end{bmatrix},$$

$$C_{(2)}^{m}=\begin{bmatrix}-7 & 4.6 & 4.5 & -5.3 & 2\\ -4.3 & -6.1 & -3.5 & 0.2 & 0\\ -5.7 & 2.6 & -3.1 & -0.2 & -2\\ -2.8 & 0.5 & 4.3 & -4.6 & 0\end{bmatrix},$$

$$C_{(2)}^{M}=\begin{bmatrix}-3 & 7.1 & 6.3 & -2.7 & 3\\ -1.7 & -2.8 & -1.5 & 2 & 0\\ -4 & 4.9 & -3.1 & 3.2 & -1\\ -0.6 & 2.5 & 6.2 & -2.8 & 0\end{bmatrix}.$$

One of the robustly $\mathcal{D}$-stabilizing gains is obtained by K=[– 29.7979 17.3083 36.4956 141.3820].

Example 2: Consider the $\mathcal{D}$-stabilization problem of the continuous-time polytope of matrices given by

$$A_1=\begin{bmatrix}-1 & 1\\ -1 & -1\end{bmatrix},\ A_2=\begin{bmatrix}-2 & 1\\ -1 & 1\end{bmatrix},$$
$$B_1=\begin{bmatrix}1\\ -1\end{bmatrix},\ B_2=\begin{bmatrix}-1\\ 2\end{bmatrix}.$$

The polytope of matrices defined by the two vertices A_1, A_2 is not stable. The region of pole placement is the intersection of a disk centered at $\alpha_1=-0.4$ with radius r = 1, a conic sector defined by its inner angle $2\theta(\theta=\pi/3)$ and its apex $\alpha_2=-0.25$, and a left half-plane defined for $\alpha_3=-0.75$.

This example is from [10] where a robustly $\mathcal{D}$-stabilizing gain is given by K=[– 0.0809 – 0.3849] using the conic complementarity algorithm. Applying Theorem 4, simplified as a total-vertex-based condition without uncertainties, we are able to find that this $\mathcal{D}$-stabilization problem is feasible. In detail, the $\mathcal{D}$-region in this example is characterized by

$$\mathcal{D}=\mathcal{D}_1\cap\mathcal{D}_2\cap\mathcal{D}_3,$$
$$\mathcal{D}_1=\left\{z\in\mathrm{C}:\alpha_1^2-r^2-\alpha_1 z-\alpha_1 z^*+zz^*<0\right\},$$
$$\mathcal{D}_2=\left\{\begin{array}{l}z\in\mathrm{C}:\\ \begin{bmatrix}\begin{pmatrix}\sin\theta\ (-2\alpha_2\\ +z+z^*)\end{pmatrix} & \cos\theta\left(z-z^*\right)\\ \cos\theta\left(z^*-z\right) & \begin{pmatrix}\sin\theta\ (-2\alpha_2\\ +z+z^*)\end{pmatrix}\end{bmatrix}<0\end{array}\right\},$$
$$\mathcal{D}_3=\left\{z\in\mathrm{C}:-2\alpha_3+z+z^*<0\right\}.$$

Applying Theorem 4 to $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ respectively with the common variables G and Z, the corresponding LMIs are feasible when the free variables are prescribed as $\varepsilon_1=1.02$ and $\varepsilon_2=\varepsilon_3=0.5$. One of the robustly $\mathcal{D}$-stabilizing gains is obtained by $K=\left[-0.1679\quad -0.3738\right]$.

6. CONCLUSIONS

This paper has presented sufficient conditions for the quadratic $\mathcal{D}$-stability and further robust $\mathcal{D}$-stability of dynamic interval systems. A parameter-dependent Lyapunov function is introduced into the robust $\mathcal{D}$-stability analysis to reduce the conservatism of quadratic $\mathcal{D}$-stability condition. This robust $\mathcal{D}$-stability condition is in terms of partial-vertex-based interval matrices other than the total vertex matrices of dynamic interval systems. The results are also extended to the $\mathcal{D}$-stabilization problem for any LMI $\mathcal{D}$-region. All the proposed conditions can be simplified to a set of LMIs and thus the approach in this paper leads to a less computational complexity than that in previous literature.

REFERENCES

[1] M. Mansour, "Robust stability of interval matrices," *Proc. of the 28th IEEE Conf. on Decision and Control*, pp. 46-51, 1989.

[2] M. E. Sezer and D. D. Siljak, "On stability of interval matrices," *IEEE Trans. on Automatic Control*, vol. 39, no. 2, pp. 368-371, 1994.

[3] K. Wang, A. N. Michel, and D. Liu, "Necessary and sufficient conditions for the Hurwitz and Schur stability of interval matrices," *IEEE Trans. on Automatic Control*, vol. 39, no. 6, pp. 1251-1255, 1994.

[4] J. Rohn, "An algorithm for checking stability of symmetric interval matrices," *IEEE Trans. on Automatic Control*, vol. 41, no. 1, pp. 133-136, 1996.

[5] Y. Xiao and R. Unbehauen, "Robust Hurwitz and Schur stability test for interval matrices," *Proc. of the 39th IEEE Conf. on Decision and Control*, pp. 4209-4214, 2000.

[6] R. K. Yedavalli, "A necessary and sufficient 'extreme point' solution for checking robust stability of interval matrices," *Proc. of American Control Conference*, pp. 1893-1897, 1999.

[7] R. K. Yedavalli, "An improved extreme point solution for checking robust stability of interval matrices with much reduced vertex set and combinatorial effort," *Proc. of American Control Conference*, pp. 3902-3907, 2001.

[8] D. Peaucelle, D. Arzelier, O. Bachelier, and J. Bernussou, "A new robust D-stability condition for real convex polytopic uncertainty," *Systems & Control Letters*, vol. 40, no. 1, pp. 21-30, 2000.

[9] V. J. S. Leite and P. L. D. Peres, "An improved LMI condition for robust D-stability of uncertain polytopic systems," *IEEE Trans. on Automatic Control*, vol. 48, no. 3, pp. 500-504, 2003.

[10] D. Arzelier, D. Henrion, and D. Peaucelle, "Robust D stabilization of a polytope of matrices," *International Journal of Control*, vol. 75, no. 10, pp. 744-752, 2002.

Wei-Jie Mao was born in 1969. He received the B.E. and Ph.D. degrees from Zhejiang University, China, in 1991 and 1996, respectively. He is currently a Professor of Zhejiang University. His research interests include robust control, time-delay systems, decoupling control theory and applications.

Jian Chu was born in 1963. He received the B.E. and M.S. degrees from Zhejiang University in 1982 and 1984, respectively. From 1986 to 1989, he studied in Kyoto University, Japan, and finished the Ph.D thesis there. Then, he came back to Zhejiang University for postdoctoral research. He is currently a Professor of Zhejiang University. He is interested in control theory and applications, R&D of computer control systems and APC software.