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# Observer-Based Mixed $H_2/H_\infty$ Control Design for Linear Systems with Time-Varying Delays: An LMI Approach

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**Abstract:** This paper presents a convex optimization method for observer-based mixed $H_2/H_\infty$ control design of linear systems with time-varying state, input and output delays. Delay-dependent sufficient conditions for the design of a desired observer-based control are given in terms of linear matrix inequalities (LMIs). An observer-based controller which guarantees asymptotic stability and a mixed $H_2/H_\infty$ performance for the closed-loop system of the linear system with time-varying delays is then developed. A Lyapunov-Krasovskii method underlies the observer-based mixed $H_2/H_\infty$ control design. A numerical example with simulation results illustrates the effectiveness of the methodology.

**Keywords:** LMI, mixed $H_2/H_\infty$ control, observer-based control, time-delay.

## 1. INTRODUCTION

Delay differential systems represent a class of infinite-dimensional systems and are assuming an increasingly important role in many disciplines like economics, biology, chemistry, mechanics, viscoelasticity, physics, physiology, population dynamics, mathematics as well as in engineering sciences, biosystems, underwater vehicles and so on (see for instance the references [1-5], and the references therein). For instance, in many control systems, delays appear either in the state, in the control input, or in the measurements. The presence of a delay in a system may be the result of some essential simplification of the corresponding process model. Therefore, the delay effects problem on the stability of systems including delays in the state and/or the input is a problem of recurring interest since the delay presence may induce complex behaviors (oscillation, instability, bad performances) for the system (see the references [3,6-8]). Among the past results on delay systems, the LMI approach is an efficient method to solve many filtering and control problems such as stability analysis and stabilization (see for instance the references [9-16]), $H_\infty$ filter design (see for instance the references [17-23]), $H_\infty$ control problems (see for instance the references [24-30]), and guaranteed-cost (observer-based) control (see for instance the references [31-37]). On the other hand, in spite of the fact that $H_\infty$ controllers are robust with respect to the disturbances since they use no statistical information, they are conservative. The multiobjective control designs are quite useful for robust performance design of systems under parameter perturbations and uncertain disturbances (see, e.g., [38]). A recent work that employs robust mixed $H_2/H_\infty$ delayed state-feedback control for a class of neutral systems with time-varying discrete and distributed delays in state has been completed in the reference [39].

On the other hand, the problem of observer design for reconstructing state variables is a more involved issue in systems with any kind of delay. In general, some sufficient conditions for the existence of an observer have been established and computational algorithms for construction of the observers have been presented in the literature, see for instance the references [40-42]. In [43], the spectrum assignment method was introduced to design the state observer. Lyapunov stability theory is used to design the state observers for linear time-varying or nonlinear systems, see the references [41-42] and [44]. In recent years, the study of mixed $H_2/H_\infty$ filters (or the so-called cost guaranteed filters) has gained growing interest; see the references [34-36] and [45]. A new LMI-based approach to design mixed $H_2/H_\infty$ filters for both discrete- and continuous-time systems with polytopic

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bounded parameters are presented, respectively, in the references [35] and [36]. Recently, problem of guaranteed-cost observer-based control was studied in the reference [32] for a class of uncertain neutral time-delay systems such that the convex optimization problem is formulated in terms of LMIs and an equality constraint which are not in the classis LMI solvable form. An LMI-based approach to design the observer-based control was developed in the reference [46] for a class of linear systems with state perturbations such the convergence rate of the system is estimated. However, the mixed $H_2/H_\infty$ performance and delay-dependent robust stability are not investigated for a time delay system in these works. Up to now, to the best of our knowledge, no results about the delay-dependent observer-based mixed $H_2/H_\infty$ control for linear systems with time-varying state, input and output delays which are in the classis LMI solvable form are available in the literature and remains to be important and challenging. This motivates the present study.

In this paper, we are concerned to develop an efficient convex optimization approach for delay-dependent observer-based mixed $H_2/H_\infty$ state feedback control problem of linear systems with time-varying state, input and output delays. Unlike the references [32] and [46], the main merit of the proposed method is the fact that it provides a convex problem such the observer and the control gains can be found from the LMI formulations without any equality constraint. Then, new required sufficient conditions are established in terms of delay-dependent LMIs combined with the Lyapunov-Krasovskii method for the existence of the desired delay-dependent observer-based mixed $H_2/H_\infty$ control such that the resulting observer error system is asymptotically stable and satisfies $H_2$ performance measure with a guaranteed cost and a prescribed level of $H_\infty$ performance measure, simultaneously. A numerical example is given to illustrate the use of our results.

This paper is organized as follows. Section 2 states the problem formulation and the needed assumptions and definitions. Section 3 includes main results of the paper which are sufficient conditions to design an observer-based controller. Section 4 provides an illustrative example. Finally, Section 5 concludes the paper.

The notations used throughout the paper are fairly standard. $I$ and 0 represent identity matrix and zero matrix; the superscript $'T'$ stands for matrix transposition; $\Re^n$ denotes the $n$-dimensional Euclidean space; $\Re^{n\times m}$ is the set of all real $m$ by $n$ matrices. $\|.\|$ refers to the Euclidean vector norm or the induced matrix 2-norm. $col\{\cdots\}$ and $diag\{\cdots\}$ represent, respectively, a column vector and a block diagonal matrix. The notation $P>0$ means that $P$ is real symmetric and positive definite; the symbol $*$ denotes the elements below the main diagonal of a symmetric block matrix. $tr(A)$ is trace of the matrix $A$. In addition, $L_2[0,\infty)$ is the space of square-integrable vector functions over $[0,\infty)$. Matrices, if the dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

## 2. PROBLEM DESCRIPTION

Consider a class of linear systems with discrete delays in the state, input and output as

$$\begin{aligned}\dot{x}(t) = A_0\, x(t) + A_1\, x(t-h(t)) + B_0\, u(t) \\ + B_1\, u(t-\eta(t)) + E_0\, w(t),\end{aligned} \tag{1a}$$

$$x(t) = \phi(t), \quad t\in[-\max\{h_M,\eta_M\},\, 0], \tag{1b}$$

$$\begin{aligned}z(t) = C_0\, x(t) + C_1\, x(t-h(t)) + D_0\, u(t) \\ + D_1\, u(t-\eta(t)),\end{aligned} \tag{1c}$$

$$y(t) = C_2\, x(t) + C_3\, x(t-h(t)) + E_1\, w(t), \tag{1d}$$

where $x(t)\in\Re^n$ is the state vector; $u(t)\in\Re^r$ is the control vector; $w(t)\in\Re^q$ is the vector of external excitations (disturbances), $z(t)\in\Re^s$ is the vector of controlled outputs and $y(t)\in\Re^p$ is the vector of measured outputs. The coefficient matrices $A_0, A_1, B_0, B_1, E_0, E_1, \{C_i\}_{i=0}^3, D_0$ and $D_1$ are real matrices with appropriate dimensions. $\phi(t)$ is a time-varying vector valued initial function, and the time-varying delays $h(t)$ and $\eta(t)$ are functions satisfying, respectively,

$$\begin{aligned}0<h(t)\le h_M, \qquad \dot{h}(t)\le h_D<1,\\ 0<\eta(t)\le \eta_M, \qquad \dot{\eta}(t)\le \eta_D<1.\end{aligned} \tag{2}$$

**Remark 1:** The system (1) with time-varying delays (2) considers the case that the derivatives of the time-varying delays $h(t)$ and $\eta(t)$ shall be less than one, which are conservative constraints. According to the references [47,48] using Leibniz-Newton formula, i.e., $x(t) = x(t-h(t)) + \int_{t-h(t)}^{t}\dot{x}(s)ds$, and some free weighting matrices, this restriction is removed, i.e., $\dot{h}(t)\le h_D$ or $\dot{\eta}(t)\le\eta_D$, which means that a fast time-varying delay is allowed.

In this paper, we focus on the design of a full order observer-based mixed $H_2/H_\infty$ control of the form

$$\dot{\hat{x}}(t) = A_0 \hat{x}(t) + A_1 \hat{x}(t-h(t)) + B_0 u(t) + B_1 u(t-\eta(t)) + L(y(t) - C_2 \hat{x}(t) + C_3 \hat{x}(t-h)), \quad (3a)$$

$$\hat{x}(t) = 0, \qquad t \in [-\max\{h_M, \eta_M\}, 0], \quad (3b)$$

$$\hat{z}(t) = C_0 \hat{x}(t) + C_1 \hat{x}(t-h(t)) + D_0 u(t) + D_1 u(t-\eta(t)), \quad (3c)$$

$$u(t) = K \hat{x}(t), \quad (3d)$$

where $\hat{x}(t) \in \Re^n$ is the observer state vector, $K$ and $L$ are the controller and observer gain matrices. By defining $e(t) = x(t) - \hat{x}(t)$ as the error vector and using the Leibniz-Newton formula instead of the time-delayed terms, then we obtain the following state-space model, namely observer error system, for the observer-based control system (1)-(3) by

$$\dot{X}(t) = \tilde{A}_1 X(t) + \tilde{A}_2 \int_{t-h(t)}^{t} \dot{X}(s)\, ds + \tilde{B}_1 \tilde{K} \int_{t-\eta(t)}^{t} \dot{X}(s)\, ds + \tilde{E} w(t) \quad (4a)$$

$$X(t) = \begin{bmatrix} \phi(t)^T & \phi(t)^T \end{bmatrix}^T, \ t \in \begin{bmatrix} -\max\{h_M, \eta_M\}, & 0 \end{bmatrix}, \quad (4b)$$

$$z(t) - \hat{z}(t) = \bar{C}_0 X(t) + \bar{C}_1 \int_{t-h(t)}^{t} \dot{X}(s)\, ds \quad (4c)$$

where

$$X(t) = col\{x(t), e(t)\}, \ \bar{A}_0 = A_0 + A_1 + (B_0 + B_1)K,$$
$$\bar{A}_1 = A_0 + A_1 - L(C_2 + C_3), \ \bar{A}_2 = -A_1 + LC_3,$$
$$\bar{B}_0 = -(B_0 + B_1)K, \ \bar{E}_0 = E_0 - LE_1,$$
$$\bar{C}_0 = [0 \quad C_0 + C_1], \ \bar{C}_1 = -[0 \quad C_1],$$
$$\tilde{K} := diag\{K, K\}, \ \tilde{A}_1 = \begin{bmatrix} \bar{A}_0 & \bar{B}_0 \\ 0 & \bar{A}_1 \end{bmatrix},$$
$$\tilde{A}_2 = \begin{bmatrix} -A_1 & 0 \\ 0 & \bar{A}_2 \end{bmatrix}, \ \tilde{B}_1 = \begin{bmatrix} -B_1 & B_1 \\ 0 & 0 \end{bmatrix}, \ \tilde{E} = \begin{bmatrix} E_0 \\ \bar{E}_0 \end{bmatrix}.$$

**Definition 1:**

(i) The $H_2$ performance measure of the system (1)-(3) is defined as

$$J_2 = \int_0^\infty [x^T(t) S_1 x(t) + e^T(t) S_2 e(t) + u^T(t) S_3 u(t)] dt,$$

where $w(t) \equiv 0$ and constant matrices $\{S_i\}_{i=1}^3 > 0$ are given.

(ii) The $H_\infty$ performance measure of the system (1)-(3) is defined as

$$J_\infty = \int_0^\infty [(z(t) - \hat{z}(t))^T (z(t) - \hat{z}(t)) - \gamma^2 w^T(t) w(t)] dt,$$

where the positive scalar $\gamma$ is given.

The problem to be addressed in this paper is formulated as follows: given the linear system (1) with time-varying delays (2) and a prescribed level of disturbance attenuation $\gamma > 0$, find a mixed $H_2/H_\infty$ state-feedback control $u(t)$ of the form $u(t) = K(x(t) - e(t))$ such that

1) the observer error system (4) is asymptotically stable for any time delays satisfying (2);
2) under $w(t) \equiv 0$, the $H_2$ performance measure guarantees $J_2 \le J_0$, where the positive scalar $J_0$ is said to be a guaranteed cost;
3) under zero initial conditions and for all non-zero $w(t) \in L_2[0, \infty)$, the $H_\infty$ performance measure guarantees $J_\infty < 0$ (or induced $L_2$-norm of the operator form $w(t)$ to the controlled outputs $z(t)$ is less than $\gamma$);

in this case, the linear system (1) with the observer-based control (3) is said to be robustly asymptotically stable with a mixed $H_2/H_\infty$ performance.

## 3. MAIN RESULTS

In this section, sufficient conditions for the solvability of the robust observer-based mixed $H_2/H_\infty$ control design problem are proposed using the Lyapunov method and an LMI approach is developed. We present first delay-dependent conditions of both $H_2$ and $H_\infty$ performance measures for the robust stability analysis of the observer error system (4) for any time-varying delays satisfying (2) in the following theorem.

**Theorem 1:** For given scalars $h_M, \eta_M > 0$, $h_D$, $\eta_D < 1$ and $\gamma > 0$, the observer error system (4) with any time-varying delays (2) is robustly stabilizable by (3d) and satisfies both $H_2$ and $H_\infty$ performance measures in the sense of Definition 1, if there exist some positive-definite matrices $\{P_i\}_{i=1}^2$ and $\{Q_i\}_{i=1}^3$, such that the following matrix inequalities are feasible,

$$\Pi_1 = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \begin{bmatrix} P_1 & 0 \\ * & P_2 \end{bmatrix} \tilde{E} & \bar{C}_0^T \\ * & \Sigma_{22} & 0 & 0 & \bar{C}_1^T \\ * & * & \Sigma_{33} & 0 & 0 \\ * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & -I \end{bmatrix} + \begin{bmatrix} \tilde{A}_1^T \\ \tilde{A}_2^T \\ \tilde{B}_1^T \\ \tilde{E}^T \\ 0 \end{bmatrix} \tilde{K} \begin{bmatrix} \tilde{A}_1^T \\ \tilde{A}_2^T \\ \tilde{B}_1^T \\ \tilde{E}^T \\ 0 \end{bmatrix}^T < 0, \quad (5a)$$

$$\Pi_2 = \begin{bmatrix} \hat{\Sigma}_{11} & \Sigma_{12} & \Sigma_{13} \\ * & \Sigma_{22} & 0 \\ * & * & \Sigma_{33} \end{bmatrix} + \begin{bmatrix} \tilde{A}_1^T \\ \tilde{A}_2^T \\ \tilde{B}_1^T \end{bmatrix} \tilde{K} \begin{bmatrix} \tilde{A}_1^T \\ \tilde{A}_2^T \\ \tilde{B}_1^T \end{bmatrix}^T < 0. \quad (5b)$$

with

$$\Sigma_{11} = \begin{bmatrix} P_1 & 0 \\ * & P_2 \end{bmatrix} \tilde{A}_1 + \tilde{A}_1^T \begin{bmatrix} P_1 & 0 \\ * & P_2 \end{bmatrix} + h_D \begin{bmatrix} Q_1 & 0 \\ * & P_2 \end{bmatrix},$$

$$\hat{\Sigma}_{11} = \Sigma_{11} + \begin{bmatrix} S_1 & 0 \\ * & S_2 \end{bmatrix} + \bar{K}^T \begin{bmatrix} S_3 & -S_3 \\ * & S_3 \end{bmatrix} \bar{K},$$

$$\Sigma_{33} = -\frac{(1-\eta_D)}{\eta_M} \begin{bmatrix} Q_2 & 0 \\ * & Q_3 \end{bmatrix},$$

$$\Sigma_{13} = \begin{bmatrix} P_1 & 0 \\ * & P_2 \end{bmatrix} \tilde{B}_1,$$

$$\Sigma_{22} = -\frac{(1-h_D)(1+h_M)}{h_M} \begin{bmatrix} Q_1 & 0 \\ * & P_2 \end{bmatrix},$$

$$\Sigma_{12} = \begin{bmatrix} P_1 & 0 \\ * & P_2 \end{bmatrix} \tilde{A}_2 + (1-h_D) \begin{bmatrix} Q_1 & 0 \\ * & P_2 \end{bmatrix},$$

and $\tilde{K} = h_M \begin{bmatrix} Q_1 & 0 \\ * & P_2 \end{bmatrix} + \eta_M \bar{K}^T \begin{bmatrix} Q_2 & 0 \\ * & Q_3 \end{bmatrix} \bar{K}.$

Moreover, an upper bound of the $H_2$ performance measure is obtained by

$$\begin{aligned} J_0 &= \phi(0)^T (P_1 + P_2) \phi(0) \\ &+ \int_{-h(0)}^{0} \phi(s)^T (Q_1 + P_2) \phi(s) ds \\ &+ \int_{-h(0)}^{0} (s + h(0)) \dot{\phi}(s)^T (Q_1 + P_2) \dot{\phi}(s) ds \\ &+ \int_{-\eta(0)}^{0} (s + \eta(0)) \dot{\phi}(s)^T K^T (Q_2 + Q_3) K \dot{\phi}(s) ds. \end{aligned} \quad (6)$$

**Proof:** Define the Lyapunov-Krasovskii functional

$$V(t) = \sum_{i=1}^{4} V_i(t), \quad (7)$$

where

$$V_1(t) = X(t)^T \begin{bmatrix} P_1 & 0 \\ * & P_2 \end{bmatrix} X(t),$$

$$V_2(t) = \int_{t-h(t)}^{t} X(s)^T \begin{bmatrix} Q_1 & 0 \\ * & P_2 \end{bmatrix} X(s) ds,$$

$$V_3(t) = \int_{t-h(t)}^{t} (s - t + h(t)) \dot{X}(s)^T \begin{bmatrix} Q_1 & 0 \\ * & P_2 \end{bmatrix} \dot{X}(s) ds,$$

$$V_4(t) = \int_{t-\eta(t)}^{t} (s - t + \eta(t)) \dot{X}(s)^T \bar{K}^T \begin{bmatrix} Q_2 & 0 \\ * & Q_3 \end{bmatrix} \bar{K} \dot{X}(s) ds,$$

with $0 < P_i = P_i^T$ for $i = 1,2$ and $0 < Q_i = Q_i^T$ for $i = 1, \cdots, 3$.

Differentiating $V_1(t)$ along the system trajectories (4) becomes

$$\begin{aligned} \dot{V}_1(t) &= 2X(t)^T \begin{bmatrix} P_1 & 0 \\ * & P_2 \end{bmatrix} \dot{X}(t) = 2X(t)^T \begin{bmatrix} P_1 & 0 \\ * & P_2 \end{bmatrix} \\ &\times \{\tilde{A}_1 X(t) + \tilde{A}_2 \int_{t-h(t)}^{t} \dot{X}(s) ds \\ &+ \tilde{B}_1 \bar{K} \int_{t-\eta(t)}^{t} \dot{X}(s) ds + \tilde{E} w(t)\}. \end{aligned}$$

Also, differentiating the second Lyapunov term in (7) gives

$$\begin{aligned} \dot{V}_2(t) &= X(t)^T \begin{bmatrix} Q_1 & 0 \\ * & P_2 \end{bmatrix} X(t) - (1 - \dot{h}(t)) X(t - h(t))^T \\ &\times \begin{bmatrix} Q_1 & 0 \\ * & P_2 \end{bmatrix} X(t - h(t)) \le X(t)^T \begin{bmatrix} Q_1 & 0 \\ * & P_2 \end{bmatrix} X(t) \\ &- (1 - h_D) X(t - h(t))^T \begin{bmatrix} Q_1 & 0 \\ * & P_2 \end{bmatrix} X(t - h(t)) \end{aligned}$$

and using the Leibniz-Newton formula the expression above is written as

$$\begin{aligned} \dot{V}_2(t) &\le h_D X(t)^T \begin{bmatrix} Q_1 & 0 \\ * & P_2 \end{bmatrix} X(t) - (1 - h_D) \\ &\times (\int_{t-h(t)}^{t} \dot{x}(s)^T ds) \begin{bmatrix} Q_1 & 0 \\ * & P_2 \end{bmatrix} (\int_{t-h(t)}^{t} \dot{x}(s) ds) \\ &+ 2(1 - h_D) X(t)^T \begin{bmatrix} Q_1 & 0 \\ * & P_2 \end{bmatrix} (\int_{t-h(t)}^{t} \dot{x}(s) ds). \end{aligned}$$

By using the inequality

$$\rho \int_0^{\rho} \varphi(s)^T Z \varphi(s) ds \ge (\int_0^{\rho} \varphi(s) ds)^T Z (\int_0^{\rho} \varphi(s) ds)$$

for any constant positive-definite matrix Z, scalar $\rho > 0$ and vector function $\varphi(.)$ [49], the time derivatives of the last two terms of $V(t)$ in (7) are, respectively,

$$\begin{aligned} \dot{V}_3(t) &= h(t) \dot{X}(t)^T \begin{bmatrix} Q_1 & 0 \\ * & P_2 \end{bmatrix} \dot{X}(t) - (1 - \dot{h}(t)) \\ &\times \int_{t-h(t)}^{t} \dot{X}(s)^T \begin{bmatrix} Q_1 & 0 \\ * & P_2 \end{bmatrix} \dot{X}(s) ds \\ &\le h(t) \dot{X}(t)^T \begin{bmatrix} Q_1 & 0 \\ * & P_2 \end{bmatrix} \dot{X}(t) - \frac{(1 - \dot{h}(t))}{h(t)} \end{aligned}$$

$$\times(\int_{t-h(t)}^{t}\dot{X}(s)ds)^T\begin{bmatrix} Q_1 & 0 \\ * & P_2 \end{bmatrix}(\int_{t-h(t)}^{t}\dot{X}(s)ds)$$
$$\le h_M\dot{X}(t)^T\begin{bmatrix} Q_1 & 0 \\ * & P_2 \end{bmatrix}\dot{X}(t)-\frac{(1-h_D)}{h_M}$$
$$\times(\int_{t-h(t)}^{t}\dot{X}(s)ds)^T\begin{bmatrix} Q_1 & 0 \\ * & P_2 \end{bmatrix}(\int_{t-h(t)}^{t}\dot{X}(s)ds),$$

and

$$\dot{V}_4(t)\le\eta_M\,\dot{X}(t)^T\,\bar{K}^T\begin{bmatrix} Q_2 & 0 \\ * & Q_3 \end{bmatrix}\bar{K}\,\dot{X}(t)-\frac{(1-\eta_D)}{\eta_M}$$
$$\times(\int_{t-\eta(t)}^{t}\dot{X}(s)ds)^T\,\bar{K}^T\begin{bmatrix} Q_2 & 0 \\ * & Q_3 \end{bmatrix}$$
$$\bar{K}(\int_{t-\eta(t)}^{t}\dot{X}(s)ds).$$

Using the obtained derivative terms above, we obtain the following result for $\dot{V}(t)$

$$\begin{aligned}\dot{V}(t)&=\sum_{i=1}^{4}\dot{V}_i(t)\\
&\le 2X(t)^T\begin{bmatrix} P_1 & 0 \\ * & P_2 \end{bmatrix}\{\tilde{A}_1\,X(t)+\tilde{A}_2\int_{t-h(t)}^{t}\dot{X}(s)ds\\
&+\tilde{B}_1\,\bar{K}\int_{t-\eta(t)}^{t}\dot{X}(s)ds+\tilde{E}\,w(t)\}\\
&+h_D X(t)^T\begin{bmatrix} Q_1 & 0 \\ * & P_2 \end{bmatrix}X(t)\\
&-(1-h_D)(\int_{t-h(t)}^{t}\dot{x}(s)^T ds)\\
&\quad\begin{bmatrix} Q_1 & 0 \\ * & P_2 \end{bmatrix}(\int_{t-h(t)}^{t}\dot{x}(s)ds)\\
&+2(1-h_D)X(t)^T\begin{bmatrix} Q_1 & 0 \\ * & P_2 \end{bmatrix}(\int_{t-h(t)}^{t}\dot{x}(s)ds)\\
&+h_M\dot{X}(t)^T\begin{bmatrix} Q_1 & 0 \\ * & P_2 \end{bmatrix}\dot{X}(t)\\
&-\frac{(1-h_D)}{h_M}(\int_{t-h(t)}^{t}\dot{X}(s)ds)^T\\
&\quad\begin{bmatrix} Q_1 & 0 \\ * & P_2 \end{bmatrix}(\int_{t-h(t)}^{t}\dot{X}(s)ds)\\
&+\eta_M\dot{X}(t)^T\,\bar{K}^T\begin{bmatrix} Q_2 & 0 \\ * & Q_3 \end{bmatrix}\bar{K}\dot{X}(t)\\
&-\frac{(1-\eta_D)}{\eta_M}(\int_{t-\eta(t)}^{t}\dot{X}(s)ds)^T\\
&\times\bar{K}^T\begin{bmatrix} Q_2 & 0 \\ * & Q_3 \end{bmatrix}\bar{K}(\int_{t-\eta(t)}^{t}\dot{X}(s)ds).\end{aligned}\tag{8}$$

Now, to establish the $H_\infty$ performance measure for the system (1), under zero initial conditions, then we have $V(t)|_{t=0}=0$. Consider the index $J_\infty$ in Definition 1, then along the solution of (4) for any nonzero $w(t)$ there holds

$$\begin{aligned}J_\infty&\le\int_0^\infty[(z(t)-\hat{z}(t))^T(z(t)-\hat{z}(t))\\
&-\gamma^2w^T(t)w(t)]dt-V(t)|_{t=0}+V(t)|_{t=\infty}\\
&\le\int_0^\infty[(z(t)-\hat{z}(t))^T(z(t)-\hat{z}(t))\\
&-\gamma^2w^T(t)w(t)+\dot{V}(t)]dt.\end{aligned}\tag{9}$$

Substituting the equivalent term of $z(t)-\hat{z}(t)$ in (4c) and the right hand side of $\dot{V}(t)$ in (8) results in (9) being less than the integrand $\vartheta(t)^T\,\Pi_1\,\vartheta(t)$ where the matrix $\Pi_1$ is given in (5a) and the vector $\vartheta(t)$ is

$$\begin{aligned}\vartheta(t):=col\{X(t),\int_{t-h(t)}^{t}\dot{X}(s)ds,\\
\int_{t-\eta(t)}^{t}\bar{K}\dot{X}(s)ds,w(t)\}.\end{aligned}\tag{10}$$

Now, if $\Pi_1<0$, then $J_\infty<0$ which means that the $L_2$ -gain from the disturbance $w(t)$ to the controlled output $z(t)$ is less than $\gamma$.

On the other hand, by applying the same Lyapunov-Krasovskii functional candidate (7), the index $J_2$ of the observer error system (4) under $w(t)\equiv 0$ can be written as

$$\begin{aligned}J_2&=\int_0^\infty[x^T(t)S_1\,x(t)+e^T(t)S_2e(t)\\
&+(x(t)-e(t))^T\times K^TS_3K(x(t)-e(t))]dt\\
&\le\int_0^\infty[x^T(t)(S_1+K^TS_3K)x(t)\\
&+e^T(t)(S_2+K^TS_3K)e(t)\\
&-2x^T(t)K^TS_3\,Ke(t)+\dot{V}(t)]dt\\
&\le\int_0^\infty\hat{\vartheta}^T(t)\Pi_2\hat{\vartheta}(t)dt,\end{aligned}\tag{11}$$

where $\vartheta(t):=col\{X(t),\int_{t-h(t)}^{t}\dot{X}(s)ds,\int_{t-\eta(t)}^{t}\bar{K}\dot{X}(s)ds\}$ and the matrix $\Pi_2$ is given in (5b). Therefore, the condition $\Pi_2<0$ in (11) implies

$$\begin{aligned}\dot{V}(t)\le-x^T(t)(S_1+K^TS_3K)x(t)-e^T(t)(S_2\\
+K^TS_3K)e(t)+2x^T(t)K^TS_3K\,e(t)\end{aligned}\tag{12}$$

or equivalently,

$$\int_0^{\infty} \dot{V}(t)dt = \lim_{t\to\infty} V(t) - V(0)$$
$$\le -\int_0^{\infty} [x^T(t) S_1 x(t) + e^T(t) S_2 e(t) \quad (13)$$
$$+ \hat{x}^T(t) K^T S_3 K \hat{x}(t)]dt.$$

It follows from (12) that $\dot{V}(t) \le -\mu \| x(t) \|^2$ for some sufficient small $\mu > 0$. Therefore, the observer error system (4) is asymptotically stable (see the reference [50]). Now, by considering the asymptotically stability of the system (1) by (3d) the $H_2$ performance measure for the observer error system (4) is established as

$$\int_0^{\infty} [x^T(t) S_1 x(t) + e^T(t) S_2 e(t) + \hat{x}^T(t) K^T S_3 K \hat{x}(t)]dt \le V(0) = J_0, \quad (14)$$

where $J_0$ is given by (6). □

**Remark 2:** Note that the cross term involving the state and the error vector in the Lyapunov-Krasovskii functional (7) has not been considered in this paper, and this structure is enforced in order to remove some nonlinearity terms in the matrix computations. However, we will study its convexity and try to find its decupling technique in our future work.

**Lemma 1:** For given positive-definite matrices $\Gamma$ and $\Theta$, the nonlinear matrix inequality $K^T \Theta^{-1} K < \Gamma^{-1}$ is satisfied if the LMI holds

$$\begin{bmatrix} -2X + \Gamma & W^T \\ * & -\Theta \end{bmatrix} < 0, \quad (15)$$

with $W := KX$.

**Proof:** In view of Schur Complement with (15), we obtain the matrix inequality

$$-2X + \Gamma + W^T \Theta^{-1} W < 0. \quad (16)$$

Using the equality

$$(X - \Gamma)\Gamma^{-1}(X - \Gamma) = X\Gamma^{-1}X - 2X + \Gamma \ge 0,$$

for any positive-definite matrix $\Gamma$ and replacing $W = KX$ in (16), it can be guaranteed that (15) implies $K^T \Theta^{-1} K < \Gamma^{-1}$. □

Now, we are in a position to give our main results on the existence of a delay-dependent observer-based mixed $H_2/H_\infty$ control in the form of (3), and show how to construct the desired control for the linear system (1).

**Theorem 2:** Consider the linear system (1) with time-varying delays (2). For given scalars $h_M, \eta_M > 0$, $h_D, \eta_D < 1$ and $\gamma > 0$, there exits a delay-dependent observer-based mixed $H_2/H_\infty$ control in the form of (3) such that the resulting observer error system (4) is robustly asymptotically stable and satisfies a mixed $H_2/H_\infty$ performance in the sense of Definition 1, if there exist matrices $\{W_i\}_{i=1}^2$ and positive-definite matrices $X_1, P_2, \{\bar{Q}_i\}_{i=1}^3$ and $\Gamma_1$, satisfying the following LMIs

$$\begin{bmatrix}
\tilde{\Sigma}_{11} & \tilde{\Sigma}_{12} & \tilde{\Sigma}_{37}^T & \tilde{\Sigma}_{14} & \bar{C}_0^T & \begin{bmatrix} h_D X_1 \\ 0 \end{bmatrix} & \tilde{\Sigma}_{17} & \tilde{\Sigma}_{17} & \begin{bmatrix} 0 \\ -\phi_2 \end{bmatrix} & 0 \\
* & \tilde{\Sigma}_{22} & 0 & 0 & \bar{C}_1^T & 0 & \tilde{\Sigma}_{27} & \tilde{\Sigma}_{27} & 0 & I \\
* & * & \tilde{\Sigma}_{33} & 0 & 0 & 0 & \tilde{\Sigma}_{37} & \tilde{\Sigma}_{37} & 0 & 0 \\
* & * & * & -\gamma^2 I & 0 & 0 & \tilde{\Sigma}_{47} & \tilde{\Sigma}_{47} & 0 & 0 \\
* & * & * & * & -I & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -h_D \bar{Q}_1 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & \tilde{\Sigma}_{77} & 0 & \begin{bmatrix} -\phi_2 \\ 0 \end{bmatrix} & 0 \\
* & * & * & * & * & * & * & \tilde{\Sigma}_{88} & \begin{bmatrix} -\phi_2 \\ 0 \end{bmatrix} & 0 \\
* & * & * & * & * & * & * & * & -X_1 & 0 \\
* & * & * & * & * & * & * & * & * & -X_1
\end{bmatrix} < 0, \quad (17a)$$

$$\begin{bmatrix}
\tilde{\Sigma}_{11} + \begin{bmatrix} 0 & 0 \\ * & S_2 \end{bmatrix} & \tilde{\Sigma}_{12} & \tilde{\Sigma}_{37}^T \\
* & \tilde{\Sigma}_{22} & 0 \\
* & * & \tilde{\Sigma}_{33} \\
* & * & * \\
* & * & * \\
* & * & * \\
* & * & * \\
* & * & *
\end{bmatrix}$$

$$
\begin{bmatrix}
\begin{bmatrix} h_D X_1 \\ 0 \end{bmatrix} & \begin{bmatrix} X_1 \\ 0 \end{bmatrix} & \begin{bmatrix} W_1^T \\ -W_1^T \end{bmatrix} & \tilde{\Sigma}_{17} & \tilde{\Sigma}_{17} & \begin{bmatrix} 0 \\ -\phi_2 \end{bmatrix} & 0 \\
0 & 0 & 0 & \tilde{\Sigma}_{27} & \tilde{\Sigma}_{27} & 0 & I \\
0 & 0 & 0 & \tilde{\Sigma}_{37} & \tilde{\Sigma}_{37} & 0 & 0 \\
-h_D\bar{Q}_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & -S_1^{-1} & 0 & 0 & 0 & 0 & 0 \\
* & * & .-S_3^{-1} & 0 & 0 & 0 & 0 \\
* & * & * & \tilde{\Sigma}_{77} & 0 & \begin{bmatrix} -\phi_2 \\ 0 \end{bmatrix} & 0 \\
* & * & * & * & \tilde{\Sigma}_{88} & \begin{bmatrix} -\phi_2 \\ 0 \end{bmatrix} & 0 \\
* & * & * & * & * & -X_1 & 0 \\
* & * & * & * & * & * & -X_1
\end{bmatrix}
< 0, \quad (17b)
$$

$$
\begin{bmatrix} -2X_1 + \Gamma_1 & W_1^T \\ * & -\bar{Q}_2 \end{bmatrix} < 0, \qquad (17c)
$$

$$
\begin{bmatrix} -2X_1 + P_2 & W_1^T \\ * & -\bar{Q}_3 \end{bmatrix} < 0, \qquad (17d)
$$

where

$$
\phi_1 := (A_0 + A_1)X_1, \; \phi_2 := (B_0 + B_1)W_1,
$$

$$
\phi_3 := P_2(A_0 + A_1), \; \phi_4 := -W_2(C_2 + C_3),
$$

$$
\phi_5 := -P_2 A_1 + W_2 C_3, \; \tilde{\Sigma}_{11} = \tilde{\Sigma}_{17} + \tilde{\Sigma}_{17}^T + \begin{bmatrix} 0 & 0 \\ * & h_D P_2 \end{bmatrix},
$$

$$
\tilde{\Sigma}_{12} = \begin{bmatrix} -A_1\bar{Q}_1 + (1-h_D)X_1 & 0 \\ * & \phi_5 + (1-h_D)P_2 \end{bmatrix},
$$

$$
\tilde{\Sigma}_{14} = \begin{bmatrix} E_0 \\ P_2E_0 - W_2E_1 \end{bmatrix}, \; \tilde{\Sigma}_{17} = \begin{bmatrix} \phi_1^T + \phi_2^T & 0 \\ 0 & \phi_3^T + \phi_4^T \end{bmatrix},
$$

$$
\tilde{\Sigma}_{27} = \begin{bmatrix} -\bar{Q}_1 A_1^T & 0 \\ 0 & \phi_5^T \end{bmatrix},
$$

$$
\tilde{\Sigma}_{22} = -(1-h_D)\begin{bmatrix} (h_M^{-1}+1)\bar{Q}_1 & 0 \\ * & 2P_2 \end{bmatrix},
$$

$$
\tilde{\Sigma}_{33} = -(1-\eta_D)\eta_M^{-1}\begin{bmatrix} \bar{Q}_2 & 0 \\ * & \bar{Q}_3 \end{bmatrix}, \; \tilde{\Sigma}_{37} = \begin{bmatrix} -\bar{Q}_2 B_1^T & 0 \\ \bar{Q}_3 B_1^T & 0 \end{bmatrix},
$$

$$
\tilde{\Sigma}_{47} = \begin{bmatrix} E_0^T & E_0^T P_2 - E_1^T W_2^T \end{bmatrix}, \; \tilde{\Sigma}_{77} = -h_M^{-1}\begin{bmatrix} \bar{Q}_1 & 0 \\ * & P_2 \end{bmatrix},
$$

and $\tilde{\Sigma}_{88} = -\eta_M^{-1}\begin{bmatrix} \Gamma_1 & 0 \\ * & P_2 \end{bmatrix}$.

The desired observer and control gains in (3) are given by

$$
K = W_1 X_1^{-1} \text{ and } L = P_2^{-1} W_2 \text{ from LMIs (17),} \qquad (18)
$$

and an upper bound of the $H_2$ performance measure is obtained by

$$
\begin{aligned}
J_0 &= \phi(0)^T (X_1^{-1} + P_2)\phi(0) \\
&+ \int_{-h(0)}^{0} \phi(s)^T (\bar{Q}_1^{-1} + P_2)\phi(s)ds \\
&+ \int_{-h(0)}^{0} (s + h(0))\dot{\phi}(s)^T (\bar{Q}_1^{-1} + P_2)\dot{\phi}(s)ds \\
&+ \int_{-\eta(0)}^{0} (s + \eta(0))\dot{\phi}(s)^T K^T (\bar{Q}_2^{-1} + \bar{Q}_3^{-1})K\dot{\phi}(s)ds.
\end{aligned} \qquad (19)
$$

**Proof:** Let

$$
\zeta = diag\{X_1, \underbrace{I, \cdots, I}_{7\ elements}\}, \qquad (20)
$$

where $X_1 = P_1^{-1}$. Premultiplying $\zeta$ and postmultiplying $\zeta^T$ to (5a) lead to

$$
\begin{bmatrix}
\Xi_{11} + \Xi_{11}^T + \begin{bmatrix} h_D X_1 Q_1 X_1 & \bar{B}_0 \\ * & h_D P_2 \end{bmatrix} & \Xi_{12} & \tilde{B}_1 & \begin{bmatrix} E_0 \\ P_2\bar{E}_0 \end{bmatrix} & \bar{C}_0^T \\
* & \Xi_{22} & 0 & 0 & \bar{C}_1^T \\
* & * & \Sigma_{33} & 0 & 0 \\
* & * & * & -\gamma^2 I & 0 \\
* & * & * & * & -I
\end{bmatrix}
+ \begin{bmatrix} \hat{\Xi}_{11}^T \\ \tilde{A}_2^T \\ \tilde{B}_1^T \\ \tilde{E}^T \\ 0 \end{bmatrix} \tilde{K} \begin{bmatrix} \hat{\Xi}_{11}^T \\ \tilde{A}_2^T \\ \tilde{B}_1^T \\ \tilde{E}^T \\ 0 \end{bmatrix}^T < 0 \quad (21)
$$

with

$$
\Xi_{11} = \begin{bmatrix} X_1\bar{A}_0^T & 0 \\ * & \bar{A}_1^T P_2 \end{bmatrix}, \; \hat{\Xi}_{11} = \begin{bmatrix} \bar{A}_0 X_1 & \bar{B}_0 \\ 0 & \bar{A}_1 \end{bmatrix},
$$

$$
\Xi_{12} = \begin{bmatrix} -A_1 + (1-h_D)X_1 Q_1 & 0 \\ * & P_2\bar{A}_2 + (1-h_D)P_2 \end{bmatrix},
$$

and $\Xi_{22} = -(1-h_D)\begin{bmatrix} (h_M^{-1}+1)Q_1 & 0 \\ * & 2P_2 \end{bmatrix}$.

By Lemma 1 and considering $W_1 := KX_1$ in LMIs (17c) and (17d), it is easily seen that the following matrix inequality can be satisfied

$$\bar{K}^T \begin{bmatrix} \bar{Q}_2^{-1} & 0 \\ * & \bar{Q}_3^{-1} \end{bmatrix} \bar{K} < \begin{bmatrix} \Gamma_1^{-1} & 0 \\ * & P_2^{-1} \end{bmatrix}, \qquad (22)$$

where $\bar{Q}_2 := Q_2^{-1}$ and $\bar{Q}_3 := Q_3^{-1}$. From (22) and applying the Schur complement to the matrix inequality (21), we obtain

$$\begin{bmatrix} \Xi_{11} + \Xi_{11}^T + \begin{bmatrix} 0 & \bar{B}_0 \\ * & h_D P_2 \end{bmatrix} & \Xi_{12} & \tilde{B}_1 & \begin{bmatrix} E_0 \\ P_2 \bar{E}_0 \end{bmatrix} & \bar{C}_0^T & \begin{bmatrix} h_D X_1 \\ 0 \end{bmatrix} & \hat{\Xi}_{11}^T & \hat{\Xi}_{11}^T \\ * & \Xi_{22} & 0 & 0 & \bar{C}_1^T & 0 & \tilde{A}_2^T & \tilde{A}_2^T \\ * & * & \Sigma_{33} & 0 & 0 & 0 & \tilde{B}_1^T & \tilde{B}_1^T \\ * & * & * & -\gamma^2 I & 0 & 0 & \tilde{E}^T & \tilde{E}^T \\ * & * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & * & -h_D \bar{Q}_1 & 0 & 0 \\ * & * & * & * & * & * & \hat{\Xi}_{77} & 0 \\ * & * & * & * & * & * & * & \hat{\Xi}_{88} \end{bmatrix} < 0$$

with

$$\hat{\Xi}_{77} = -h_M^{-1} \begin{bmatrix} \bar{Q}_1 & 0 \\ * & P_2^{-1} \end{bmatrix}, \quad \hat{\Xi}_{88} = -\eta_M^{-1} \begin{bmatrix} \Gamma_1 & 0 \\ * & P_2^{-1} \end{bmatrix}.$$

Again, applying the congruence transformation $diag\{I, I, \bar{Q}_1, I, \bar{Q}_2, \bar{Q}_3, \underbrace{I, \cdots, I}_{4\ elements}, P_2, I, P_2\}$ with $\bar{Q}_1 := Q_1^{-1}$ to the matrix inequality above implies

$$\begin{bmatrix} \Xi_{11} + \Xi_{11}^T + \begin{bmatrix} 0 & 0 \\ * & h_D P_2 \end{bmatrix} & \hat{\Xi}_{12} & \tilde{\Sigma}_{37}^T & \begin{bmatrix} E_0 \\ P_2 \bar{E}_0 \end{bmatrix} & \bar{C}_0^T & \begin{bmatrix} h_D X_1 \\ 0 \end{bmatrix} & \Xi_{11} & \Xi_{11} \\ * & \Xi_{22} & 0 & 0 & \bar{C}_1^T & 0 & \hat{\Xi}_{27} & \hat{\Xi}_{27} \\ * & * & \tilde{\Sigma}_{33} & 0 & 0 & 0 & \tilde{\Sigma}_{37} & \tilde{\Sigma}_{37} \\ * & * & * & -\gamma^2 I & 0 & 0 & \begin{bmatrix} E_0^T & \bar{E}_0^T P_2 \end{bmatrix} & \begin{bmatrix} E_0^T & \bar{E}_0^T P_2 \end{bmatrix} \\ * & * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & * & -h_D \bar{Q}_1 & 0 & 0 \\ * & * & * & * & * & * & \tilde{\Sigma}_{77} & 0 \\ * & * & * & * & * & * & * & \tilde{\Sigma}_{88} \end{bmatrix}$$

$$+ \begin{bmatrix} \begin{bmatrix} 0 \\ -(B_0 + B_1) \end{bmatrix} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \begin{bmatrix} -(B_0 + B_1) \\ 0 \end{bmatrix} \\ \begin{bmatrix} -(B_0 + B_1) \\ 0 \end{bmatrix} \end{bmatrix} K \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} K^T \begin{bmatrix} \begin{bmatrix} 0 \\ -(B_0 + B_1) \end{bmatrix} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \begin{bmatrix} -(B_0 + B_1) \\ 0 \end{bmatrix} \\ \begin{bmatrix} -(B_0 + B_1) \\ 0 \end{bmatrix} \end{bmatrix}^T < 0, \qquad (23)$$

where

$$\hat{\Xi}_{12} = \begin{bmatrix} -A_1 \bar{Q}_1 + (1 - h_D) X_1 & 0 \\ * & P_2 \bar{A}_2 + (1 - h_D) P_2 \end{bmatrix},$$

$$\hat{\Xi}_{27} = \begin{bmatrix} -\bar{Q}_1 A_1^T & 0 \\ 0 & \bar{A}_2^T P_2 \end{bmatrix}.$$

Using the inequality $X^T Y + Y^T X \le X^T \Omega X + Y^T \Omega^{-1} Y$ for any matrices $X, Y$ and a positive-definite matrix $\Omega = \Omega^T > 0$, we have (by consider-ing $\Omega = X_1$):

$$\begin{bmatrix} \begin{bmatrix} 0 \\ -(B_0 + B_1) \end{bmatrix} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \begin{bmatrix} -(B_0 + B_1) \\ 0 \end{bmatrix} \\ \begin{bmatrix} -(B_0 + B_1) \\ 0 \end{bmatrix} \end{bmatrix} K \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} K^T \begin{bmatrix} \begin{bmatrix} 0 \\ -(B_0 + B_1) \end{bmatrix} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \begin{bmatrix} -(B_0 + B_1) \\ 0 \end{bmatrix} \\ \begin{bmatrix} -(B_0 + B_1) \\ 0 \end{bmatrix} \end{bmatrix}^T$$

$$\leq \begin{bmatrix} \begin{bmatrix} 0 \\ -(B_0+B_1) \end{bmatrix} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \begin{bmatrix} -(B_0+B_1) \\ 0 \end{bmatrix} \\ \begin{bmatrix} -(B_0+B_1) \\ 0 \end{bmatrix} \end{bmatrix} KX_1K^T \begin{bmatrix} \begin{bmatrix} 0 \\ -(B_0+B_1) \end{bmatrix} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \begin{bmatrix} -(B_0+B_1) \\ 0 \end{bmatrix} \\ \begin{bmatrix} -(B_0+B_1) \\ 0 \end{bmatrix} \end{bmatrix}^T + \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} X_1^{-1} \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T . \tag{24}$$

From (23)-(24) and using Schur Complement and considering the term of $KX_1K^T$ in (24) as $(KX_1)X_1^{-1}(KX_1)^T$, the following matrix inequality is easily obtained

$$\begin{bmatrix} \Xi_{11}+\Xi_{11}^T+\begin{bmatrix} 0 & 0 \\ * & h_DP_2 \end{bmatrix} & \hat{\Xi}_{12} & \tilde{\Sigma}_{37}^T & \begin{bmatrix} E_0 \\ P_2\bar{E}_0 \end{bmatrix} & \bar{C}_0^T & \begin{bmatrix} h_DX_1 \\ 0 \end{bmatrix} & \Xi_{11} & \Xi_{11} & \begin{bmatrix} 0 \\ \hat{B} \end{bmatrix} & 0 \\ * & \Xi_{22} & 0 & 0 & \bar{C}_1^T & 0 & \hat{\Xi}_{27} & \hat{\Xi}_{27} & 0 & I \\ * & * & \tilde{\Sigma}_{33} & 0 & 0 & 0 & \tilde{\Sigma}_{37} & \tilde{\Sigma}_{37} & 0 & 0 \\ * & * & * & -\gamma^2 I & 0 & 0 & \begin{bmatrix} E_0^T & \bar{E}_0^T P_2 \end{bmatrix} & \begin{bmatrix} E_0^T & \bar{E}_0^T P_2 \end{bmatrix} & 0 & 0 \\ * & * & * & * & -I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -h_D\bar{Q}_1 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \tilde{\Sigma}_{77} & 0 & \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix} & 0 \\ * & * & * & * & * & * & * & \tilde{\Sigma}_{88} & \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix} & 0 \\ * & * & * & * & * & * & * & * & -X_1 & 0 \\ * & * & * & * & * & * & * & * & * & -X_1 \end{bmatrix} < 0, \tag{25}$$

where $\hat{B} := -(B_0+B_1)KX_1$.

Similarly, applying the congruence transformation $\zeta = diag\{X_1, \underbrace{I, \cdots, I}_{5\ elements}\}$ to the matrix inequality (5b) in Theorem 1 implies

$$\begin{bmatrix} \Xi_{11}+\Xi_{11}^T+\tilde{\Xi}_{11} & \Xi_{12} & \tilde{B}_1 \\ * & \Xi_{22} & 0 \\ * & * & \Sigma_{33} \end{bmatrix} + \begin{bmatrix} \hat{\Xi}_{11}^T \\ \tilde{A}_2^T \\ \tilde{B}_1^T \end{bmatrix} \tilde{K} \begin{bmatrix} \hat{\Xi}_{11}^T \\ \tilde{A}_2^T \\ \tilde{B}_1^T \end{bmatrix}^T < 0, \tag{26}$$

where

$$\tilde{\Xi}_{11} = \begin{bmatrix} X_1K^T \\ -X_1K^T \end{bmatrix} S_3 \begin{bmatrix} X_1K^T \\ -X_1K^T \end{bmatrix}^T + \begin{bmatrix} X_1(h_DQ_1+S_1)X_1 & \bar{B}_0 \\ * & h_DP_2+S_2 \end{bmatrix}.$$

Again, by Schur Complement and using the matrix inequality (22), the matrix inequality (26) yields

$$\begin{bmatrix} \Xi_{11}+\Xi_{11}^T+\begin{bmatrix} 0 & \bar{B}_0 \\ * & S_2+h_DP_2 \end{bmatrix} & \Xi_{12} & \tilde{B}_1 & \begin{bmatrix} h_DX_1 \\ 0 \end{bmatrix} & \begin{bmatrix} X_1 \\ 0 \end{bmatrix} & \begin{bmatrix} X_1K^T \\ -X_1K^T \end{bmatrix} & \hat{\Xi}_{11}^T & \hat{\Xi}_{11}^T \\ * & \Xi_{22} & 0 & 0 & 0 & 0 & \tilde{A}_2^T & \tilde{A}_2^T \\ * & * & \Sigma_{33} & 0 & 0 & 0 & \tilde{B}_1^T & \tilde{B}_1^T \\ * & * & * & -h_D\bar{Q}_1 & 0 & 0 & 0 & 0 \\ * & * & * & * & -S_1^{-1} & 0 & 0 & 0 \\ * & * & * & * & * & -S_3^{-1} & 0 & 0 \\ * & * & * & * & * & * & \hat{\Xi}_{77} & 0 \\ * & * & * & * & * & * & * & \hat{\Xi}_{88} \end{bmatrix} < 0.$$

Finally, applying the congruence transformation $diag\{I, I, \bar{Q}_1, I, \bar{Q}_2, \bar{Q}_3, \underbrace{I, \cdots, I}_{4\ elements}, P_2, I, P_2\}$ to the matrix inequality above implies

$$\begin{bmatrix} \Xi_{11}+\Xi_{11}^T+\begin{bmatrix} 0 & 0 \\ * & S_2+h_D P_2 \end{bmatrix} & \hat{\Xi}_{12} & \tilde{\Sigma}_{37}^T & \begin{bmatrix} h_D X_1 \\ 0 \end{bmatrix} & \begin{bmatrix} X_1 \\ 0 \end{bmatrix} & \begin{bmatrix} X_1 K^T \\ -X_1 K^T \end{bmatrix} & \Xi_{11} & \Xi_{11} & \begin{bmatrix} 0 \\ \hat{B} \end{bmatrix} & 0 \\ * & \Xi_{22} & 0 & 0 & 0 & 0 & \hat{\Xi}_{27} & \hat{\Xi}_{27} & 0 & I \\ * & * & \tilde{\Sigma}_{33} & 0 & 0 & 0 & \tilde{\Sigma}_{37} & \tilde{\Sigma}_{37} & 0 & 0 \\ * & * & * & -h_D \bar{Q}_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -S_1^{-1} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & .-S_3^{-1} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \tilde{\Sigma}_{77} & 0 & \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix} & 0 \\ * & * & * & * & * & * & * & \tilde{\Sigma}_{88} & \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix} & 0 \\ * & * & * & * & * & * & * & * & -X_1 & 0 \\ * & * & * & * & * & * & * & * & * & -X_1 \end{bmatrix} < 0. \quad (27)$$

Obviously, the matrix inequalities (25) and (27) include multiplication of the matrices $K, L$ and Lyapunov matrices. In the literature, more attention has been paid to the problems having this nature (see for instance the reference [51]). In the sequel, it is shown that, by a suitable change of variables, the bilinear matrix inequalities (22) and (24) are converted into a convex programming problem written in terms of LMIs. Therefore, by introducing $W_1 := KX_1$ and $W_2 := P_2 L$ as new decision variables instead of the matrices $K$ and $L$, respectively, we obtain the LMIs (17a) and (17b). □

**Remark 3:** We note that the main purpose using the matrix inequalities (22) and (24) in deriving our delay-dependent results is to convert a nonconvex problem to a convex one which is in the classis LMI solvable form.

**Remark 4:** It is worth noting that the number of the variables to be determined in the LMIs (17) is $n(2n+p+2)+r(n+r+1)$. It is also observed that the LMIs (17) are linear in the set of matrices $\{W_i\}_{i=1}^2$, $X_1, P_2, \{\bar{Q}_i\}_{i=1}^3$ and $\Gamma_1$, and the scalar $\gamma^2$. This implies that the suboptimal solution to the problem of delay-dependent observer-based mixed $H_2/H_\infty$ control can be found by solving the following convex optimization problem

$$Min \quad \lambda$$
$$subject \ to \ LMIs \ (17) \ with \ \lambda := \gamma^2.$$

Motivated by the idea of the reference [32], minimizing the upper bound of the $H_2$ performance measure is stated. Before proceeding further, we consider the following notations.

$$\begin{aligned} &\int_{-h(0)}^{0} \phi(s)\phi(s)^T ds = \aleph_1 \aleph_1^T, \\ &\int_{-h(0)}^{0} \dot{\phi}(s)\dot{\phi}(s)^T ds = \aleph_2 \aleph_2^T, \\ &\int_{-\eta(0)}^{0} (s+\mu(0))\dot{\phi}(s)\dot{\phi}(s)^T ds = \aleph_3 \aleph_3^T. \end{aligned} \quad (28)$$

**Theorem 3:** Consider the linear system (1)-(2) with $H_2$ performance measure in the sense of Definition 1. Consider the following optimization problem

$$\begin{aligned} &Min \ \{\alpha + tr(Z_1) + tr(Z_2) + tr(\aleph_3^T Z_3 \aleph_3)\} \\ &subject \ to \ (i) \ LMIs \ (17) \end{aligned} \quad (29)$$

$$(ii) \quad \begin{bmatrix} -\alpha & \phi(0)^T & \phi(0)^T P_2 \\ * & -X_1 & 0 \\ * & * & -P_2 \end{bmatrix} < 0$$

$$(iii) \quad \begin{bmatrix} -Z_1 & \aleph_1^T & \aleph_1^T P_2 \\ * & -\bar{Q}_1 & 0 \\ * & * & -P_2 \end{bmatrix} < 0$$

$$(iv) \quad \begin{bmatrix} -Z_2 & \aleph_2^T & \aleph_2^T P_2 \\ * & -\bar{Q}_1 & 0 \\ * & * & -P_2 \end{bmatrix} < 0$$

$$(v) \quad \begin{bmatrix} -2X_1 & I & W_1^T \\ * & -Z_3 & 0 \\ * & * & -\bar{Q}_2 \end{bmatrix} < 0.$$

For given scalars $h_M, \eta_M > 0$, $h_D, \eta_D < 1$ and $\gamma > 0$, if the optimization problem (29) has matrices $\{W_i\}_{i=1}^2$ and positive-definite matrices $X_1, P_2, \{\bar{Q}_i\}_{i=1}^3$, $\Gamma_1$, $\{Z_i\}_{i=1}^3$, then the observer-based control law (3d) is a suboptimal delay-dependent mixed $H_2/H_\infty$ control which ensures the minimization of the $H_2$ performance measure $J_2$.

**Proof:** According to Theorem 2, (*i*) in (29) is clear. Also, by applying Schur Complement, it is easy to see that (*ii*) is equivalent

$$-\alpha + \phi(0)^T (X_1^{-1} + P_2)\phi(0) < 0. \tag{30}$$

The second term on the right-hand side in (19) can be rewritten as

$$\begin{aligned}
&\int_{-h(0)}^{0} [\phi(s)^T (\bar{Q}_1^{-1} + P_2)\phi(s)]ds \\
&= \int_{-h(0)}^{0} [tr(\phi(s)^T (\bar{Q}_1^{-1} + P_2)\phi(s))]ds \\
&= tr(\aleph_1 \aleph_1^T (\bar{Q}_1^{-1} + P_2)) \\
&= tr(\aleph_1^T (\bar{Q}_1^{-1} + P_2)\aleph_1) \\
&< tr(Z_1).
\end{aligned} \tag{31}$$

Therefore, we get

$$\aleph_1^T (\bar{Q}_1^{-1} + P_2)\aleph_1 < Z_1 \tag{32}$$

and by applying Schur Complement, the LMI (*iii*) is easily obtained. The third term on the right-hand side in (19) can be rewritten as

$$\begin{aligned}
&\int_{-h(0)}^{0} [(s + h(0))\dot{\phi}(s)^T (\bar{Q}_1^{-1} + P_2)\dot{\phi}(s)]ds \\
&= \int_{-h(0)}^{0} [tr((s + h(0))\dot{\phi}(s)^T (\bar{Q}_1^{-1} + P_2)\dot{\phi}(s))]ds \\
&= tr(\aleph_2^T (\bar{Q}_1^{-1} + P_2)\aleph_2) \\
&< tr(Z_2).
\end{aligned} \tag{33}$$

Hence, we have

$$\aleph_2^T (\bar{Q}_1^{-1} + P_2)\aleph_2 < Z_2, \tag{34}$$

and by applying Schur Complement, the LMI (*iv*) is concluded. Finally, the fourth term on the right-hand side in (19) can be rewritten as

$$\begin{aligned}
&\int_{-\eta(0)}^{0} [(s + \eta(0))\dot{\phi}(s)^T K^T (\bar{Q}_2^{-1} + \bar{Q}_3^{-1})K\dot{\phi}(s)]ds \\
&= \int_{-\eta(0)}^{0} [tr((s + \eta(0))\dot{\phi}(s)^T K^T (\bar{Q}_2^{-1} + \bar{Q}_3^{-1})K\dot{\phi}(s))]ds \\
&= tr(\aleph_3^T K^T (\bar{Q}_2^{-1} + \bar{Q}_3^{-1})K\aleph_3) \\
&< tr(\aleph_3^T Z_3 \aleph_3)
\end{aligned} \tag{35}$$

then, we obtain

$$K^T (\bar{Q}_2^{-1} + \bar{Q}_3^{-1})K < Z_3. \tag{36}$$

By applying Lemma 1, the LMI (*v*) is obtained. Therefore, it follows that

$$J_0 < \alpha + tr(Z_1) + tr(Z_2) + tr(\aleph_3^T Z_3 \aleph_3). \tag{37}$$

Hence, if there exist a solution set to LMIs (29), the suboptimal observer-based mixed $H_2/H_\infty$ control $u(t) = W_1 X_1^{-1} x(t) - W_1 X_1^{-1} e(t)$ minimizes the upper bound of the $H_2$ performance measure $J_2$ of the observer error system (4). □

## 4. NUMERICAL EXAMPLE

In this section, we will verify the proposed methodology by giving an illustrative example. We solved LMIs (17) by using Matlab LMI Control Toolbox [52]. The example is given below.

Consider the unstable system given by

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} -1 & 0.4 \\ 0.2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0.5 & 0.1 \\ 0.4 & 0.6 \end{bmatrix} x(t - h(t)) \\
&\quad + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t - \eta(t)) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t), \\
x(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad t \in \begin{bmatrix} -1, & 0 \end{bmatrix}, \\
z(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 1 \end{bmatrix} x(t - h(t)) \\
&\quad + u(t) + u(t - \eta(t)), \\
y(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 1 \end{bmatrix} x(t - h(t)) + w(t),
\end{aligned} \tag{38}$$

with $S_1 = S_2 = I_2$, $S_3 = 1$ and constant delays $h, \eta$ with $h_M = \eta_M = 1$. It is required to design an observer-based control such that the closed-loop system is asymptotically stable and satisfies a mixed $H_2/H_\infty$ performance. To this end, in light of Theorem 2 with $\gamma = 0.8$ in $H_\infty$ performance measure, we solved LMIs (17) and obtained the control and the observer gains as $K = -[0.1114 \quad 0.3935]$, and

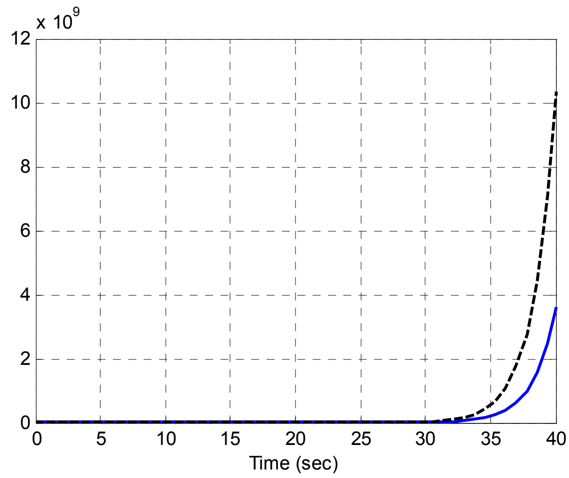


Fig. 1. Response of the open-loop system: (a) first state (solid line) and (b) second state (dashed line).

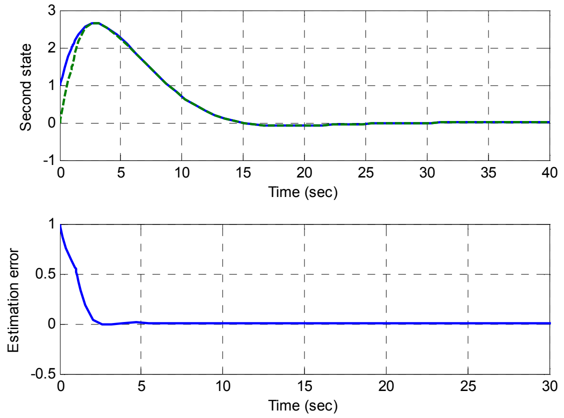


Fig. 2. Estimation results of the first state: (a) observer state (dashed line) and (b) plant state (solid line).

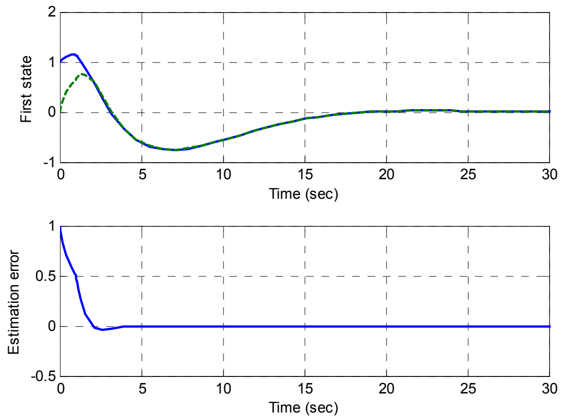


Fig. 3. Estimation results of the second state: (a) observer state (dashed line) and (b) plant state (solid line).

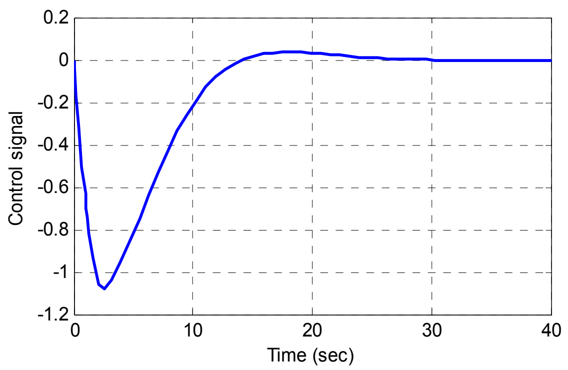


Fig. 4. Mixed $H_2/H_\infty$ control signal.

$L = [0.4530 \quad 0.6853]^T$, respectively.

For initial condition $x(0) = [1 \quad 1]$, $h = h_M$, $\eta = \eta_M$ and assuming a unit step disturbance in the time interval $[0, 1]$, the simulation results are shown in Figs. 1-4 and the corresponding suboptimal $H_2$ performance measure of the closed-loop system is $J_0 = 9.1848$. As depicted in Fig. 1, the open-loop system is unstable and the state response curves become unbounded. The controlled plant and the observer state trajectories plus their estimation errors are depicted in Figs. 2 and 3. It is observed that the observer is doing well to estimate the plant states. Finally, the curve of mixed $H_2/H_\infty$ control in (3d) is also shown in Fig. 4.

## 5. CONCLUSIONS

An observer-based mixed $H_2/H_\infty$ control design method was presented in this paper for linear systems with time-varying state, input and output delays. Delay-dependent sufficient conditions to design a desired observer-based control were given by Lyapunov-Krasovskii method in terms of LMIs. An observer-based controller guaranteeing asymptotic stability, and a mixed $H_2/H_\infty$ performance of the closed-loop system was developed. A numerical example was given to show the effectiveness of the method.

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