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Improved Delay-independent H_2 Performance Analysis and Memoryless State Feedback for Linear Delay Systems with Polytopic Uncertainties

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Abstract: An improved linear matrix inequality (LMI) representation of delay-independent H_2 performance analysis is introduced for linear delay systems with delays of any size. Based on this representation we propose a new H_2 memoryless state feedback design. By introducin g a new matrix variable, the new LMI formulation enables us to parameterize memoryles s controllers without involving the Lyapunov variables in the formulations. By using a parameter-dependent Lyapunov function, this new representation proposed here provides us the results with less conservatism.

Keywords: Bounded real lemma, controller synthesis, H_2 performance, time-delay systems commas.

1. INTRODUCTION

As is well known, H_2 performance is useful to handle stochastic aspects such as measurement noise and random disturbance. Meanwhile, robust H_2 problem is developed in the efforts to provide stability margins to the H_2 optimal (LQG) regulator in the 1970s. The difficulties encountered in the combination of the classical and modern control [1,2] led to a shift in focus to other performance criteria (H_∞, L_1), which are directly linked to robust stability guarantees by means of small gain theorem. However, robust control methods based on H_∞ and L_1 measures lean too heavily on robustness and sacrifice an adequate view of performance; the latter is often more naturally described by an H_2 performance criterion, which can be used to capture both the transient response of the system and the response to stationary noise. The promise of a successful combination of robustness and H_2 performance was renewed in the late 1980s (see [3-7]). By introducing some additional variables there are lots of literatures concerning improved H_∞ or H_2 analysis and synthesis of linear uncertain systems without delays [8-15].

As to linear delay systems, the study concerning H_2 control can be classified into two types: delay-dependent and delay-independent results. A delay-dependent H_2 controller ensures asymptotic stability and a prescribed H_2 performance for any delays smaller than a given bound, while a delay-independent H_2 controller guarantees asymptotic stability and a prescribed H_2 performance for delays of any size. However, the bound of delay is not previously known in many actual sites, including communication over the Internet. Thus, the study of delay-independent H_2 performance analysis and synthesis is very meaningful. In this paper, we will focus on delay-independent H_2 performance analysis and memoryless H_2 controller design for linear delay systems. Based on standard delay-independent H_2 performance analysis condition, H_2 performance computation problem of linear system with delays with any size can be presented as a standard LMI optimization formulation [16], which includes the product of the constant Lyapunov function matrix and system matrices.

The main conservatism of the existing delay-independent H_2 performance analysis conditions stems from the inequality bounding technique employed for some cross terms encountered in the performance analysis. By introducing some slack matrix variables, less conservative LMI representation of delay-independent H_2 analysis and synthesis conditions for linear delay systems have not been explored fully yet. It motivates the present study.

In this paper, first, an equivalent LMI representation of delay-independent H_2 performance analysis for linear delay systems is introduced. By introducing a new matrix variable, the new representation is linear with Lyapunov function matrix and system matrix and does not include any product of them. It provides us with a numerical computation method of H_2 norm. Secondly, by using parameter-

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dependent Lyapunov function; this representation can reduce the conservatism that occurs in the controller design problem with a fixed Lyapunov function. Then based on this representation, we consider robust H_2 memoryless state feedback synthesis problem. We demonstrated the applicability of the new method on two examples. And our results are compared with the standard H_2 performance analysis formulation, where a fixed Lyapunov function was used.

2. PRELIMINARY

Given the following linear continuous-time delaysystem G described in state space form by the equations

$$\dot{x}(t) = Ax(t) + A_1 x(t-\tau) + B_w w(t), \quad (1)$$

$$z(t) = C_{z0} x(t) + C_{z1} x(t-\tau). \quad (2)$$

We assume that the initial conditions are null,

$$x(t) = 0, \forall t \in [-\tau, 0]. \quad (3)$$

Matrices $(A, A_1, B_w, C_{z0}, C_{z1}, D_{zw})$ are constant matrices of appropriate dimensions. $x(t) \in \Re^n$ is system state vector, $w(t) \in \Re^q$ is exogenous disturbance signal and $z(t) \in \Re^m$ is objective function signal including state combination, τ is the delay of the system.

For a prescribed scalar $\gamma > 0$, we define H_2 performance index by

$$\|G\|_2^2 := \lim_{h\to\infty} E\left\{\frac{1}{h}\int_0^h z^T(t) z(t) dt\right\}, \quad (4)$$

when the initial conditions are null and $w(t)$ is a zero-mean white process with an identity power spectrum density matrix, where in the above E denotes mathematical expectation.

First, we will give the standard delay-independent H_2 performance analysis for these systems as follows:

Lemma 1: Consider the system (1)-(3), for a given scalar $\gamma > 0$, this system is asymptotically stable and $\|G\|_2^2 < \gamma$ for any constant delay parameter $\tau \geq 0$ if there exist symmetric and positive definite matrices P_0, P_1 and W, such that the LMI

$$\begin{bmatrix} A^T P_0 + P_0 A + P_1 & P_0 A_1 & C_{z0}^T \\ (*) & -P_1 & C_{z1}^T \\ (*) & (*) & -I \end{bmatrix} < 0, \quad (5a)$$

$$\begin{bmatrix} W & B_w^T P_0 \\ (*) & P_0 \end{bmatrix} > 0, \quad Trace(W) < \gamma, \quad (5b)$$

has a feasible solution.

The proof of this lemma can be referred to [4,16].

Remark 1: we can find that the LMI (5) is not suitable for the controller synthesis problem, using Schur complement lemma and similar transformation as to (5), by letting $P_0^{-1} = Q_0, P_1^{-1} = Q_1$, we obtain the equivalent formulation of (5) as

$$\begin{bmatrix} Q_0 A^T + A Q_0 & Q_0 C_{z0}^T & A_1 Q_1 & Q_0 \\ (*) & -I & C_{z1} Q_1 & 0 \\ (*) & (*) & -Q_1 & 0 \\ (*) & (*) & (*) & -Q_1 \end{bmatrix} < 0, \quad (6a)$$

$$\begin{bmatrix} W & B_w^T \\ (*) & Q_0 \end{bmatrix} > 0, \quad Trace(W) < \gamma. \quad (6b)$$

This LMI representation is convenient for us to analyze and synthesize nominal control performance for linear delay systems, when system matrices $(A, A_1, B_w, C_{z0}, C_{z1})$ do not include any polytopic-type uncertainties. However, in the case of linear delay systems with polytopic-type uncertainties, it will result in very conservative computation for H_2cost γ due to the constant Lyapunov function matrix. When parameter-dependent Lyapunov function is introduced to reduce conservatism in (6), this representation can not be extended to controller design problem due to the product of Lyapunov function matrix and system matrix.

3. A NEW LMI REPRESENTATION OF H_2 PERFORMANCE ANALYSIS

In this section, first we propose a new equivalent LMI representation of H_2 performance analysisfor linear delay systems with delays of any size. Then, this condition is considered to compute H_2 guaranteed cost for linear delay system with polytopic-type uncertainties.

Theorem 1: There exist symmetric positive-definite matrices Q_0, Q_1 and W to satisfy (6), if and only if there exist symmetric positive-matrices Q_0, Q_1, W and a general matrix F satisfying

$$\begin{bmatrix} AF + F^T A^T & Q_0 - F^T + rAF & F^T C_{z0}^T & A_1 Q_1 & Q_0 \\ (*) & -r(F+F^T) & rF^T C_{z0}^T & 0 & 0 \\ (*) & (*) & -I & C_{z1} Q_1 & 0 \\ (*) & (*) & (*) & -Q_1 & 0 \\ (*) & (*) & (*) & (*) & -Q_1 \end{bmatrix} < 0, \quad (7a)$$

$$\begin{bmatrix} W & B_w^T \\ (*) & Q_0 \end{bmatrix} > 0, \quad Trace(W) < \gamma \quad (7b)$$

for a sufficiently small positive scalar r.

Proof: When symmetric positive-definite matrices Q_0, Q_1 and W satisfying (6a) and (6b) exists, we always can find a positive scalar $r>0$ as $r<2\lambda_1/\lambda_2$, where

$$\lambda_1=\lambda_{\min}\left(-\begin{bmatrix} Q_0A^T+AQ_0 & Q_0C_{z0}^T & A_1Q_1 & Q_0 \\ (*) & -I & C_{z1}Q_1 & 0 \\ (*) & (*) & -Q_1 & 0 \\ (*) & (*) & (*) & -Q_1 \end{bmatrix}\right)$$

and

$$\lambda_2=\lambda_{\max}\left(\begin{bmatrix} AQ_0A^T & AQ_0C_{z0}^T & 0 & 0 \\ C_{z0}Q_0A^T & C_{z0}Q_0C_{z0}^T & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right).$$

Then applying Schur complement with respect to (7a) by choosing $F=Q_0$, we have

$$\begin{bmatrix} Q_0A^T+AQ_0 & Q_0C_{z0}^T & A_1Q_1 & Q_0 \\ (*) & -I & C_{z1}Q_1 & 0 \\ (*) & (*) & -Q_1 & 0 \\ (*) & (*) & (*) & -Q_1 \end{bmatrix} +\frac{r}{2}\begin{bmatrix} AQ_0A^T & AQ_0C_{z0}^T & 0 & 0 \\ C_{z0}Q_0A^T & C_{z0}Q_0C_{z0}^T & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}<0. \quad (8)$$

The scalar r makes (7a) always satisfy.

When positive symmetric matrices Q_0, Q_1, W, a general matrix F and a positive scalar $r>0$ satisfying (7a) exist, we multiply (7a) with

$$T=\begin{bmatrix} I & A & 0 & 0 & 0 \\ 0 & C_{z0} & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

on the left and T^T on the right, since matrix T is full row rank, we can get (6a) directly. □

Based on this formulation, we will consider the case of linear delay systems with polytopic-type uncertainties. Assuming system matrices $(A(a), A_1(a), B_w(a), C_{z0}(a), C_{z1}(a))$ are not precisely known, but belong to a polytopic uncertainty domain ∂, we have

$$(A(a),A_1(a),B_w(a),C_{z0}(a),C_{z1}(a))\in\partial =: \left\{\begin{array}{l}(A(a),A_1(a),B_w(a),C_{z0}(a),C_{z1}(a)) \\ =\sum_{i=1}^{N}a_i(A_i,A_{1,i},B_{w,i},C_{z0,i},C_{z1,i}), \\ a_i\ge 0, i=1,\dots,N, \sum_{i=1}^{N}a_i=1\end{array}\right\}, \quad (9)$$

where $(A_i,A_{1,i},B_{w,i},C_{z0,i},C_{z1,i})$, $i=1,\dots,N$, are constant matrices with appropriate dimensions, and a_i, $i=1,\dots,N$, are time-invariant uncertainties. Theorem 1 is extended to linear delay systems as (9) by employing a parameter-dependent Lyapunov function as follows:

Theorem 2: Given system (9), its H2 norm is less than a prescribed value of γ, if there exist positive symmetric matrices $Q_{0,i}$, Q_1, W and a general matrix F satisfying

$$\begin{bmatrix} A_iF+F^TA_i^T & Q_{0,i}-F^T+rA_iF & F^TC_{z0,i}^T & A_{1,i}Q_1 & Q_{0,i} \\ (*) & -r(F+F^T) & rF^TC_{z0,i}^T & 0 & 0 \\ (*) & (*) & -I & C_{z1,i}Q_1 & 0 \\ (*) & (*) & (*) & -Q_1 & 0 \\ (*) & (*) & (*) & (*) & -Q_1 \end{bmatrix}<0, \quad (10a)$$

$$\begin{bmatrix} W & B_{w,i}^T \\ (*) & Q_{0,i} \end{bmatrix}>0, \quad Trace(W)<\gamma, \quad (10b)$$

$i=1,\dots,N$, for a positive scalar r.

Thereby, H2 control performance of uncertain continuous-time systems is guaranteed with a prescribed value of γ. By introducing this parameter- dependent Lyapunov function matrix

$$Q_0(a)=\sum_{i=1}^{N}a_iQ_{0,i},\ a_i\ge 0, i=1,\dots,N, \sum_{i=1}^{N}a_i=1,$$

H2 guaranteed cost γ will be obtained less than quadratic Lyapunov function based results, where Lyapunov function matrix is a fixed one.

Since general matrix F is assumed to be constant one as to system matrices with polytopic-type uncertainties, Theorem 2 is also suitable for control synthesis purpose. Furthermore, the conditions (10) above will be used to H_2 memoryless state-feedback synthesis control problem.

Remark 2: there are some other ideas proposed in [17-19], which could be used for the problem considered in this paper for the future research. The crucial point in these papers is that no com-mon matrix variable is required for the entire uncertainty domain.

4. H_2 MEMORYLESS STATE FEEDBACK

In this section, Theorem 2 will be extended to solve H_2 memoryless state-feedback control problem for linear delay systems with polytopic-type uncertainties consider the following linear delay system:

$$\begin{aligned} \dot{x}(t) &= A(a)x(t) + A_1(a)x(t-\tau) + B_w(a)w(t) \\ &\quad + B_u(a)u(t), \qquad (11) \\ z(t) &= C_{z0}(a)x(t) + C_{z1}(a)x(t-\tau) + D_{zu}(a)u(t), \end{aligned}$$

where x, z and w are as in (1)-(3) and $u \in \Re^r$ is the control input.

Assuming that the system matrices lie with the following polytope as

$$(A(a), A_1(a), B_w(a), B_u(a), C_{z0}(a), C_{z1}(a), D_{zu}(a)) \in \partial =: \left\{ \begin{aligned} &(A(a), A_1(a), B_w(a), B_u(a), C_{z0}(a), C_{z1}(a), D_{zu}(a)) \\ &= \sum_{i=1}^{N} a_i (A_i, A_{1,i}, B_{w,i}, B_{u,i}, C_{z0,i}, C_{z1,i}, D_{zu,i}), \\ &a_i \geq 0, i = 1,\ldots,N, \sum_{i=1}^{N} a_i = 1 \end{aligned} \right\}. \qquad (12)$$

The state-feedback control problem is to find, for a prescribed scalar $\gamma > 0$, the state-feedback gain K such that the H_2 memoryless control law of $u = Kx$ guarantees an upper bound of γ to H_2 norm.

Substituting this state-feedback control law into (11), the closed-loop system can be obtained as

$$\begin{aligned} \dot{x}(t) &= (A(a) + B_u(a)K)x(t) + A_1(a)x(t-\tau) \\ &\quad + B_w(a)w(t), \qquad (13) \\ z(t) &= (C_{z0}(a) + D_{zu}(a)K)x(t) + C_{z1}(a)x(t-\tau). \end{aligned}$$

Then, a state-feedback gain K will be solved according to the following theorem.

Theorem 3: Given system (13), its H_2 norm is less than a prescribed value of γ, if there exist positive symmetric matrices $Q_{0,i}, Q_1, W$ and general matrices F, M satisfying

$$\begin{bmatrix} A_iF + F^TA_i^T + B_{u,i}M + M^TB_{u,i}^T & Q_{0,i} - F^T + rA_iF + rMB_{u,i} & F^TC_{z0,i}^T + M^TD_{zu,i}^T & A_{1,i}Q_1 & Q_{0,i} \\ (*) & -r(F+F^T) & rF^TC_{z0,i}^T + rM^TD_{zu,i}^T & 0 & 0 \\ (*) & (*) & -I & C_{z1,i}Q_1 & 0 \\ (*) & (*) & (*) & -Q_1 & 0 \\ (*) & (*) & (*) & (*) & -Q_1 \end{bmatrix} < 0, \qquad (14a)$$

$$\begin{bmatrix} W & B_{w,i}^T \\ (*) & Q_{0,i} \end{bmatrix} > 0, \quad Trace(W) < \gamma. \; i = 1,\ldots,N \qquad (14b)$$

for a positive scalar r. If the existence is affirmative, the state-feedback gain K is given by $K = MF^{-1}$.

Remark 3: It also should be noted, as to robust performance analysis and synthesis problem, the cost value γ will not be a monotonously decreasing function with the decreasing of scalar r.

In order to obtain the minimum possible γ, we consider solving (14a) and (14b) by iterating over r. Although some computation complexity is increased, less conservative results will be obtainable.

5. NUMERICAL EXAMPLES

In this section, the approaches developed above areillustrated by a simple example; All LMIs-related computations were performed with the LMI Toolbox of Matlab [20].

We consider the problem of controlling the yaw angles of a satellite system with delays. The satellite system consisting of two rigid bodies joined by a flexible link is assumed to have the state-space representation as follows:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & -f & f \\ k & -k & f & -f \end{bmatrix} x(t) \\ &+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.01 & 0 & 0 & 0 \\ 0 & -0.001 & 0 & -0.001 \end{bmatrix} x(t-\tau) \\ &+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u, \end{aligned}$$

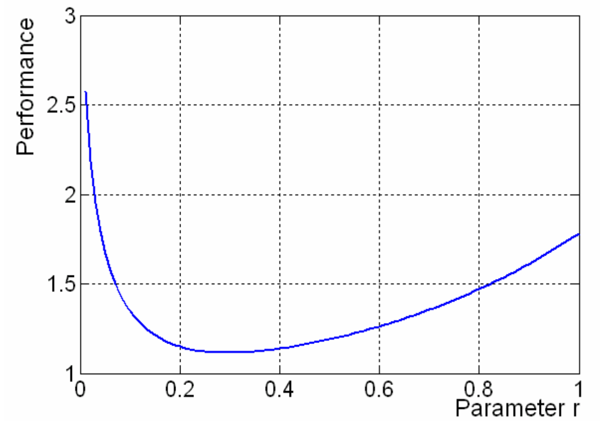

Fig. 1. The relation between performance γ and r.

$$z=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}x+\begin{bmatrix}0\\ 0.01\end{bmatrix}u,$$

where k and f are torque constant and viscous damping, which vary in the following uncertainty ranges: $k\in[0.09 \quad 0.4]$ and $f\in[0.0038 \quad 0.04]$.

Two methods are considered to solve this control problem.

1. The method of Lemma 1 with a fixed Lyapunov function matrix, the minimum guaranteed level of $\gamma=2.0335$ can be achieved with

$$K=-10^2*[0.5528 \;\; 2.601 \;\; 0.1476 \;\; 4.961].$$

2. The method of Theorem 3, the minimum guaranteedlevel of $\gamma=1.1353$ can be achieved for $r=0.22$ with state feedback gain

$$K=-10^2*[0.266 \;\; 2.268 \;\; 0.106 \;\; 2.462].$$

The relation between performance γ and r is shown as Fig. 1.

Remark 4: From above example, as to robust control synthesis problem, we can find that the cost value γ is not a monotonously decreasing function with the decreasing of scalar r, H_2 guaranteed cost $\gamma=1.1353$ is obtained for the positive scalar $r=0.22$. From above numerical examples, the method proposed in this paper provides better results than a common Lyapunov matrix based method for robust analysis and synthesis problems of H_2 control.

6. CONCLUSIONS

New equivalent LMI representations to H_2 performance analysis have been derived for linear delay systems with delays of any size. By using a parameter-dependent Lyapunov function, new representation gives us the results with less conservatism not only for H_2 norm computation but also memoryless state-feedback design of linear delay systems with polytioic-type uncertainties.

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