

# Optimal Filtering for Linear Discrete-Time Systems with Single Delayed Measurement

Hong-Guo Zhao, Huan-Shui Zhang\*, Cheng-Hui Zhang, and Xin-Min Song

**Abstract:** This paper aims to present a polynomial approach to the steady-state optimal filtering for delayed systems. The design of the steady-state filter involves solving one polynomial equation and one spectral factorization. The key problem in this paper is the derivation of spectral factorization for systems with delayed measurement, which is more difficult than the standard systems without delays. To get the spectral factorization, we apply the reorganized innovation approach. The calculation of spectral factorization comes down to two Riccati equations with the same dimension as the original systems.

**Keywords:** Diophantine equation, reorganized innovation, spectral factorization, steady-state optimal filtering.

## 1. INTRODUCTION

The optimal steady-state filtering problem for linear time-invariant plants has been well studied by two classic methods in the past decades, one is the Kalman filtering formulation [1,2], the other is the polynomial equation approach (Weiner filtering design) [3-5]. In the Kalman filtering formulation, the steady-state filter is designed by solving algebraic Riccati equation. For the polynomial equation method, the calculation of the steady-state filter involves one Diophantine equation and one spectral factorization [6-9]. However, most of the previous works for steady-state filtering focus on the delay-free systems. In the case of time-delay, the steady-state filtering for discrete-time systems can in principal be investigated via system augmentation and standard Kalman filtering (polynomial approach) [2]. However, the augmentation leads to higher dimension especially when the time-delay or system dimension is large and thus results in much expensive computation. On the other hand, we noted that the optimal filtering for time-varying systems with delayed measurement has received important progress [10,11]. Recently, [12]

has derived the finite-horizon filter by adopting the so-called reorganized innovation analysis approach. The time-varying filter is calculated by solving standard Riccati difference equations with the same dimension as the original systems. The reorganized analysis approach has been shown to be powerful to deal with some complicated problems such as  $H_\infty$  fixed-lag smoothing and so on [13].

In sprit of the reorganized innovation analysis approach developed in the previous works, this paper is to study the steady-state optimal filtering for the systems with delayed measurement based on the polynomial approach. The steady-state filter is derived by using the orthogonality between the estimation error and the observations. It is shown that the filter can be designed by performing one polynomial equation and one spectral factorization where the latter is the key problem that is to be solved in this paper. With the application of the reorganized innovation analysis approach, the spectral factorization is obtained by solving two different standard Riccati difference equations.

The rest of this paper is organized as follows. The problem to be addressed is stated in Section 2. The steady-state optimal filter is designed based on one Diophantine equation and one spectral factorization in Section 3. Section 4 presents the main results by using reorganized innovation approach. Section 5 gives a example to show the calculational procedure of steady-state optimal filter. The conclusions are drawn in Section 6.

## 2. PROBLEM STATEMENT

Consider the following linear discrete-time systems

$$x(t+1) = \Phi x(t) + \Gamma u(t), \quad 0 \leq t < \infty, \quad (1)$$

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where  $x(t) \in R^n$  and  $u(t) \in R^r$  represent the state and the systems noise, respectively. The state is observed by two different channels with delays as

$$y_{(i)}(t) = H_{(i)}x(t - d_i) + v_{(i)}(t), \quad i = 0, 1, \quad (2)$$

where the time-delays  $d_i$ ,  $i = 0, 1$  are assumed to be in increasing order as  $0 = d_0 < d_1 = d$ .  $y_{(i)}(t) \in R^{p_i}$  are delayed measurements,  $v_{(i)}(t) \in R^{p_i}$  are the measurement noises. The noises  $u(t), v_{(i)}(t)$  are mutually uncorrelated white noises with zero means and covariance matrices  $\mathcal{E}[u(k)u^T(j)] = Q_u \delta_{kj}$ ,  $\mathcal{E}[v_{(i)}(k)v_{(i)}^T(j)] = Q_{v_{(i)}} \delta_{kj}$ , where  $\delta_{kj}$  is Kronecker delta function,  $T$  stands for the transpose, and  $\mathcal{E}$  denotes the mathematical expectation.

**Assumption 1:** The above systems are completely controllable and completely observable.

In (2),  $y_{(i)}(t), i = 0, 1$  mean the observation of the state  $x(t - d_i)$  at time  $t$  with time-delay  $d_i$ . Let  $y(t)$  denote the observation of the systems (1)-(2) at time  $t$ , then we have

$$y(t) = \begin{bmatrix} y_{(0)}(t) \\ y_{(1)}(t) \end{bmatrix}. \quad (3)$$

Therefore, (2) can be rewritten as

$$y(t) = \begin{bmatrix} H_{(0)} & 0 \\ 0 & H_{(1)} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t_d) \end{bmatrix} + v_s(t), \quad (4)$$

where  $t_d = t - d$ , and  $v_s(t) = \begin{bmatrix} v_{(0)}(t) \\ v_{(1)}(t) \end{bmatrix}$  is white noise of zero mean and covariance matrix  $Q_{v_s} = \begin{bmatrix} Q_{v_{(0)}} & 0 \\ 0 & Q_{v_{(1)}} \end{bmatrix}$ .

Our problem can be stated as: Given the observation  $\{y(0), \dots, y(t)\}$ , find a steady-state optimal filter  $\hat{x}(t|t) = \bar{R}(q^{-1})y(t)$  that minimizes the following mean square error

$$\mathcal{E}[x(t) - \hat{x}(t|t)]^T [x(t) - \hat{x}(t|t)],$$

where  $\bar{R}(q^{-1})$  is a stable polynomial matrix, and  $q^{-1}$  is the backward shift operator.

### 3. STEADY-STATE OPTIMAL FILTERING

Using (1), we have

$$x(t) = (I_n - \Phi q^{-1})^{-1} \Gamma u(t-1), \quad (5)$$

where  $(I_n - \Phi q^{-1})^{-1}$  has the following form as

$$(I_n - \Phi q^{-1})^{-1} = A^{-1}(q^{-1})C(q^{-1}), \quad (6)$$

and

$$A(q^{-1}) = 1 + a_1 q^{-1} + \dots + a_{n_a} q^{-n_a},$$

$$C(q^{-1}) = C_0 + C_1 q^{-1} + \dots + C_{n_c} q^{-n_c}$$

are assumed to have no common factors, and  $A(q^{-1})$  is stable polynomial.

By substituting (6) into (5), it follows that

$$x(t) = A^{-1}(q^{-1})B(q^{-1})u(t-1), \quad (7)$$

where  $B(q^{-1}) = C(q^{-1})\Gamma$ .

On the other hand, it follows from (4) that

$$y(t) = \bar{H}(q^{-1})x(t) + v_s(t), \quad (8)$$

where  $\bar{H}(q^{-1}) = \begin{bmatrix} H_{(0)} \\ H_{(1)}q^{-d} \end{bmatrix}$ .

Substituting (7) into (8), we obtain that

$$A(q^{-1})y(t) = D(q^{-1})u(t-1) + A(q^{-1})v_s(t),$$

where

$$D(q^{-1}) = \bar{H}(q^{-1})B(q^{-1}). \quad (9)$$

**Remark 1:** Since we assume that systems (1)-(2) is completely controllable and completely observable, then  $D(q^{-1})$  and  $A(q^{-1})$  have no common factor [2].

Let

$$r(t) = D(q^{-1})u(t-1) + A(q^{-1})v_s(t),$$

which is the sum of two correlated moving average (MA) processes. We easily obtain the spectralmatrix or spectrum  $S_r(q, q^{-1})$  of  $r(t)$  as

$$S_r(q, q^{-1}) = D(q^{-1})Q_u D^T(q) + A(q^{-1})Q_{v_s} A^T(q).$$

**Assumption 2:** Spectral density matrix  $D(e^{-j\omega})Q_u D^T(e^{j\omega}) + A(e^{-j\omega})Q_{v_s} A^T(e^{j\omega})$  is positive definite matrix for all  $-\pi \leq \omega \leq \pi$ .

In view of Assumption 2, we have the following spectral factorization

$$\begin{aligned} E(q^{-1})E^T(q^1) \\ = D(q^{-1})Q_u D^T(q) + A(q^{-1})Q_{v_s} A^T(q), \end{aligned} \quad (10)$$

where  $E(q^{-1})$  is viewed as the spectral factor, which is stable.

**Remark 2:** Assumption 2 implies that there exists a unique and stable spectral factor  $E(q^{-1})$  in spectral factorization (10), see [6] and references therein.

**Remark 3:** The polynomial  $D(q^{-1})$  in (9) is not standard form studied as in [1]. One possible approach to such spectral factorization is the state augmentation which, however, results in expensive computation. In what following, we shall present one simple approach to above spectral factorization.

Firstly, we define the following sequence

$$w(t) \equiv y(t) - \hat{y}(t|t-1), \quad (11)$$

where  $w(t)$  is called the innovation associated with measurement  $\{y(0), \dots, y(t)\}$ , and  $\hat{y}(t|t-1)$  is the projection of  $y(t)$  onto the linear space  $L\{w(0), w(1), \dots, w(t-1)\}$  which is spanned by innovation sequence  $\{w(0), w(1), \dots, w(t-1)\}$ .

With the support of (4) and (11), we have that

$$y(t) = \begin{bmatrix} H_{(0)} & 0 \\ 0 & H_{(1)} \end{bmatrix} \begin{bmatrix} \hat{x}(t|t-1) \\ \hat{x}(t_d|t-1) \end{bmatrix} + w(t). \quad (12)$$

By applying the projection formula in Hilbert space,  $\hat{x}(t_d|t-1)$  is computed by

$$\hat{x}(t_d|t-1) = q^{-d} \hat{x}(t|t-1) + K(q^{-1})w(t), \quad (13)$$

where  $q^{-d} \hat{x}(t|t-1) = \hat{x}(t_d|t_d-1)$ , and  $K(q^{-1})$  is given by

$$K(q^{-1}) = \sum_{i=1}^d K_i q^{-i}, \quad (14)$$

in the above,  $K_i$  is defined as

$$K_i \equiv \varepsilon[x(t_d)w^T(t-i)]Q_w^{-1}, i=1, \dots, d, \quad (15)$$

where innovation covariance matrix  $Q_w$  is defined as

$$Q_w \equiv \varepsilon[w(t)w^T(t)]. \quad (16)$$

Substituting (13) into (12), it follows that

$$y(t) = \bar{H}(q^{-1})\hat{x}(t|t-1) + \bar{K}(q^{-1})w(t) + w(t), \quad (17)$$

where  $\bar{K}(q^{-1})$  is given as follows

$$\bar{K}(q^{-1}) = \begin{bmatrix} 0 \\ H_{(1)}K(q^{-1}) \end{bmatrix}. \quad (18)$$

On the other hand,  $\hat{x}(t|t-1)$  can be calculated by

$$\hat{x}(t|t-1) = \Phi\hat{x}(t-1|t-2) + K_0w(t-1), \quad (19)$$

where  $K_0$  is defined as

$$K_0 \equiv \varepsilon[x(t)w^T(t-1)]Q_w^{-1}. \quad (20)$$

From (19), we have

$$\hat{x}(t|t-1) = (I_n - \Phi q^{-1})^{-1} K_0 w(t-1). \quad (21)$$

Substituting (21) into (17), we obtain

$$y(t) = \bar{H}(q^{-1})(I_n - \Phi q^{-1})^{-1} K_0 w(t-1) + \bar{K}(q^{-1})w(t) + w(t). \quad (22)$$

By applying (6) and (22), we obtain the autoregressive moving average (ARMA) innovation model as

$$A(q^{-1})y(t) = \bar{H}(q^{-1})C(q^{-1})K_0 q^{-1}w(t) + \bar{K}(q^{-1})A(q^{-1})w(t) + A(q^{-1})w(t). \quad (23)$$

Note that the  $w(t)$  is the innovation, that is,  $w(t)$  is mutually uncorrelated. Then we have the following results.

**Theorem 1:** The spectral factor  $E(q^{-1})$  is computed by

$$E(q^{-1}) = \{\bar{H}(q^{-1})C(q^{-1})K_0 q^{-1} + \bar{K}(q^{-1})A(q^{-1}) + A(q^{-1})I_n\}Q_w^{1/2}, \quad (24)$$

where  $K_0$  and  $Q_w$  are defined as in (20) and (16), respectively.  $\bar{K}(q^{-1})$  can be computed by applying (18), (14) and (15).

**Proof:** By using (23), the proof is straightforward.  $\square$

Now we are in the position to present the steady-state optimal filter using Theorem 1 in the following.

**Theorem 2:** Consider the single measurement delayed systems (1)-(2). The steady-state optimal filter is formulated as

$$\hat{x}(t|t) = S(q^{-1})E^{-1}(q^{-1})y(t), \quad (25)$$

where  $S(q^{-1})$  and  $R(q^{-1})$  are solutions of the following Diophantine equation

$$B(q^{-1})Q_u B^T(q) \bar{H}^T(q) = S(q^{-1})E^T(q) + qA(q^{-1})R^T(q). \quad (26)$$

**Proof:** Considering the steady-state optimal filter  $\hat{x}(t|t)$  is the linear function of the known measurement  $\{y(0), \dots, y(t)\}$ , we suppose that  $\hat{x}(t|t)$  has the form as (25).

Then the filtering error  $\tilde{x}(t|t)$  can be given as follows

$$\tilde{x}(t|t) = x(t) - \hat{x}(t|t).$$

On the other hand, we give any admissible linear function of the measurement, i.e.,

$$\eta(t) = F(q^{-1})y(t),$$

where  $F(q^{-1})$  is any stable polynomial matrix.

It is well known that there exists the orthogonality between the filtering error  $\tilde{x}(t|t)$  and any linear function  $\eta(t)$ . Then we can obtain

$$\mathcal{E}[\tilde{x}(t|t)\eta^T(t)] = 0. \quad (27)$$

By applying (27), polynomial equation (26) can easily be obtained.  $\square$

**Remark 4:** The solvability and uniqueness of the polynomial equation (26) have been well studied in the previous works, see [14,15] for details.

**Remark 5:** Theorem 2 presents the steady-state optimal filtering for linear discrete-time measurement delayed systems. The steady-state optimal prediction problem and the fix-lag smoothing problem for time-delayed systems (1)-(2) can also be solved easily by using a similar discussion.

**Remark 6:** Although we have presented the filter in Theorem 2, the calculation for the spectral factor  $E(q^{-1})$  remains to be computed. Note that  $E(q^{-1})$  is related with  $K_i, i=0, \dots, d$  and  $Q_w$ , in what following, we shall compute these unknown polynomial matrices.

#### 4. COMPUTATION OF $K_i, i=0, \dots, d$ AND $Q_w$

In this section, our aim is to compute  $K_i, i=0, \dots, d$  and  $Q_w$ . The key technique is to employ the reorganized innovation analysis and projection approach in Hilbert space.

##### 4.1. Reorganized innovation

To calculate the  $Q_w$  and  $K_i, i=0, \dots, d$ , we firstly suppose that the systems (1)-(2) are finite-horizon. Then, the observation  $y(t)$  in (3) can be rewritten as

$$\bar{y}(t) = \begin{cases} y_{(0)}(t), & 0 \leq t < d, \\ \begin{bmatrix} y_{(0)}(t) \\ y_{(1)}(t) \end{bmatrix}, & t \geq d. \end{cases} \quad (28)$$

Next, we are to organize the current and delayed measurements and introduce a delay-free measure-

ment. It is easily known that the linear space  $L\{\bar{y}(0), \bar{y}(1), \dots, \bar{y}(t)\}$  can be equivalently written as

$$L\{y_2(s)\}_{s=0}^{t_d}; y_1(t_d+1), \dots, y_1(t)\}.$$

It is clear that  $y_1(s)$  and  $y_2(s)$  satisfy

$$y_1(s) = H_1 x(s) + v_1(s), \quad (29)$$

$$y_2(s) = H_2 x(s) + v_2(s), \quad (30)$$

where

$$H_1 = H_{(0)}, H_2 = \begin{bmatrix} H_{(0)} \\ H_{(1)} \end{bmatrix},$$

$$y_1(s) = y_{(0)}(s), y_2(s) = \begin{bmatrix} y_{(0)}(s) \\ y_{(1)}(s+d) \end{bmatrix},$$

and

$$v_1(s) = v_{(0)}(s), v_2(s) = \begin{bmatrix} v_{(0)}(s) \\ v_{(1)}(s+d) \end{bmatrix}$$

are mutually uncorrelated white noises with zero means and covariance matrices

$$Q_{v_1} = Q_{v_{(0)}}, Q_{v_2} = \begin{bmatrix} Q_{v_{(0)}} & 0 \\ 0 & Q_{v_{(1)}} \end{bmatrix}.$$

Obviously, the new measurements  $y_1(s)$  and  $y_2(s)$  are no longer with time-delay. Using the new measurements, we can introduce the following sequence

$$w(s,1) = y_1(s) - \hat{y}_1(s,1), \quad (31)$$

$$w(s,2) = y_2(s) - \hat{y}_2(s,2), \quad (32)$$

where  $\hat{y}_1(s,1)$  is the projection of  $y_1(s)$  onto the linear space  $L\{y_2(i)\}_{i=0}^{t_d}; \{y_1(i)\}_{i=t_d+1}^{s-1}$ , and  $\hat{y}_2(s)$  is the projection of  $y_2(s)$  onto the linear space  $L\{y_2(i)\}_{i=0}^{s-1}$ .

In view of (29)-(32), it follows that

$$w(s,1) = H_1 \tilde{x}(s,1) + v_1(s), \quad (33)$$

$$w(s,2) = H_2 \tilde{x}(s,2) + v_2(s), \quad (34)$$

where

$$\tilde{x}(s,1) = x(s) - \hat{x}(s,1),$$

$$\tilde{x}(s,2) = x(s) - \hat{x}(s,2),$$

where  $\hat{x}(s,1)$  and  $\hat{x}(s,2)$  are defined as in  $\hat{y}_1(s,1)$  and  $\hat{y}_2(s,2)$ .

Based on the discussion as in [12], as  $t \rightarrow +\infty$ , it

is known that  $\{w(0,2), \dots, w(t_d, 2); w(t_d + 1, 1), \dots, w(t, 1)\}$  is called the reorganized innovation sequence

Next, as  $t \rightarrow +\infty$ , considering state equation (1) and reorganized measurement (30), we introduce the following steady-state Riccati equation

$$P_2 = \Phi P_2 \Phi^T + \Gamma Q_u \Gamma^T - \Phi P_2 H_2^T Q_{w_2}^{-1} H_2 P_2 \Phi^T, \quad (35)$$

where  $P_2$  is the one-step ahead state estimation error covariance matrix.  $Q_{w_2}$  denotes the steady-state covariance matrix of  $w(\cdot, 2)$ , and can be computed by

$$Q_{w_2} = H_2 P_2 H_2^T + Q_{v_2}. \quad (36)$$

Similarly, as  $t \rightarrow +\infty$ , considering state equation (1) and reorganized measurement (29), we introduce the following steady-state Riccati equation

$$P_1(i+1) = \Phi P_1(i) \Phi^T + \Gamma Q_u \Gamma^T - \Phi P_1(i) H_1^T Q_w^{-1}(i, 1) H_1 P_1(i) \Phi^T, \quad i > 0, \quad (37)$$

where  $P_1(i)$  is the one-step ahead state estimation error covariance matrices.  $Q_w(i, 1), i > 0$  denote the steady-state covariance matrices of  $w(i, 1)$ , and can be calculated as

$$Q_w(i, 1) = H_1 P_1(i) H_1^T + Q_{v_1}, \quad i > 0. \quad (38)$$

Further, for the convenience of discussion, we now give the following definition

$$M_2(t+j, t) \equiv \mathcal{E}[x(t+j) \tilde{x}^T(t, 2)],$$

$$M_1(t+j, t+i) \equiv \mathcal{E}[x(t+j) \tilde{x}^T(t+i, 1)], \quad i > 0,$$

where

$$\tilde{x}(t, 2) = x(t) - \hat{x}(t, 2),$$

$$\tilde{x}(t+i, 1) = x(t+i) - \hat{x}(t+i, 1).$$

In the above,  $\hat{x}(t, 2)$  is the projection of  $x(t)$  onto the linear space  $L\{\{w(s, 2)\}_{s=0}^{t-1}\}$ , and  $\hat{x}(t+i, 1)$  is the projection of  $x(t+i)$  onto the linear space  $L\{\{w(s, 2)\}_{s=0}^{t-1}; \{w(s, 1)\}_{s=t+1}^{t+i-1}\}$ .

As  $t \rightarrow +\infty$ ,  $M_2(t+j, t)$  and  $M_1(t+j, t+i)$  will be independent of the time  $t$ , which is rewritten as  $M_2(j, 0)$  and  $M_1(j, i)$ .

According to [12],  $M_2(j, 0)$  and  $M_1(j, i)$  can be calculated as, respectively

$$M_2(j, 0) = \begin{cases} P_2 [A_2^T]^{-j}, & j \leq 0, \\ \Phi^j P_2 & j > 0, \end{cases} \quad (39)$$

$$M_1(j, i) = \begin{cases} P_1(j) A_1^T(j) \cdots A_1^T(i-1), & i \geq j \\ \Phi^{j-i} P_1(i), & i < j, \end{cases} \quad (40)$$

where

$$A_2 = \Phi - \Phi P_2 H_2^T Q_{w_2}^{-1} H_2,$$

$$A_1(i) = \Phi - \Phi P_1(i) H_1^T Q_w^{-1}(i, 1) H_1, \quad i > 0,$$

$P_2$  and  $P_1(i), i > 0$  are calculated by (35) and (37).  $Q_w(i, 1), i > 0$  and  $Q_{w_2}$  are calculated via (38) and (36).

#### 4.2. Solutions to the $Q_w$ and $K_i, i = 0, \dots, d$

Firstly, the innovation covariance matrix will be given in the following theorem, which is derived based on the reorganized innovation approach. The calculation does not require the augmented systems.

**Theorem 3:** The steady-state innovation covariance matrix  $Q_w$  is computed by

$$Q_w = \begin{bmatrix} H_{(0)} M_1(d, d) H_{(0)}^T + Q_{v_{(0)}} & H_{(0)} [M_1(0, d)]^T H_{(1)}^T \\ H_{(1)} M_1(0, d) H_{(0)}^T & H_{(1)} P H_{(1)}^T + Q_{v_{(1)}} \end{bmatrix}, \quad (41)$$

where

$$P = P_2 - M_2(0, 0) H_2^T Q_{w_2}^{-1} H_2 [M_2(0, 0)]^T - \sum_{i=1}^{d-1} M_1(0, i) H_1^T Q_w^{-1}(i, 1) H_1 [M_1(0, i)]^T,$$

$M_1(\cdot, \cdot)$  and  $Q_w(i, 1), i = 1, \dots, d$  are calculated via (40) and (38), respectively.  $P_2$  is given by (35).

**Proof:** From (11), we get

$$w(t) = \begin{bmatrix} H_{(0)} & 0 \\ 0 & H_{(1)} \end{bmatrix} \begin{bmatrix} x(t) - \hat{x}(t|t-1) \\ x(t_d) - \hat{x}(t_d|t-1) \end{bmatrix} + v_s(t). \quad (42)$$

Note that  $\hat{x}(t_d|t-1)$  is the projection of state  $x(t_d)$  onto the linear space  $L\{\{w(i, 2)\}_{i=0}^{t_d}; \{w(i, 1)\}_{i=t_d+1}^{t-1}\}$ , by making use of projection formula,  $\hat{x}(t_d|t-1)$  can be calculated as

$$\begin{aligned} & \hat{x}(t_d|t) \\ &= \text{Pr } \text{oj} \{x(t_d) | w(0, 2), \dots, w(t_d, 2); \\ & \quad w(t_d + 1, 1), \dots, w(t-1, 1)\} \\ &= \hat{x}(t_d, 2) + \mathcal{E}[x(t_d) w^T(t_d, 2)] Q_{w_2}^{-1} w(t_d, 2) \end{aligned} \quad (43)$$

$$\begin{aligned}
& + \sum_{i=1}^{d-1} \varepsilon[x(t_d)w^T(t_d+i,1)]Q_w^{-1}(i,1)w(t_d+i,1) \\
& = \hat{x}(t_d,2) + M_2(0,0)H_2^T Q_w^{-1} w(t_d,2) \\
& \quad + \sum_{i=1}^{d-1} M_1(0,i)H_1^T Q_w^{-1}(i,1)w(t_d+i,1).
\end{aligned}$$

At the same time, note that

$$\hat{x}(t|t-1) = \hat{x}(t,1). \quad (44)$$

By substituting (43) and (44) into (42), the innovation  $w(t)$  allows us to write

$$w(t) = \begin{bmatrix} H_{(0)} & 0 \\ 0 & H_{(1)} \end{bmatrix} \begin{bmatrix} \tilde{x}(t,1) \\ \zeta(t) \end{bmatrix} + v_s(t), \quad (45)$$

where

$$\tilde{x}(t,1) = x(t) - \hat{x}(t,1),$$

and

$$\begin{aligned}
\zeta(t) & = x(t_d) - \hat{x}(t_d|t-1) \\
& = \tilde{x}(t_d,2) - M_2(0,0)H_2^T Q_w^{-1} w(t_d,2) \\
& \quad - \sum_{i=1}^{d-1} M_1(0,i)H_1^T Q_w^{-1}(i,1)w(t_d+i,1).
\end{aligned}$$

Then the innovation covariance matrix  $Q_w$  is given by

$$\begin{aligned}
Q_w & = \begin{bmatrix} H_{(0)} & 0 \\ 0 & H_{(1)} \end{bmatrix} \\
& \quad \times \begin{bmatrix} \varepsilon[\tilde{x}(t,1)\tilde{x}^T(t,1)] & \varepsilon[\tilde{x}(t,1)\zeta^T(t)] \\ \varepsilon[\zeta(t)\tilde{x}^T(t,1)] & \varepsilon[\zeta(t)\zeta^T(t)] \end{bmatrix} \\
& \quad \times \begin{bmatrix} H_{(0)} & 0 \\ 0 & H_{(1)} \end{bmatrix}^T + Q_{v_s}.
\end{aligned} \quad (46)$$

In view of (40), it follows

$$\varepsilon[\tilde{x}(t,1)\tilde{x}^T(t,1)] = M_1(d,d). \quad (47)$$

Also, by considering the fact that  $\zeta(t)$  is uncorrelated with  $w(t_d+i,1)$ ,  $i=1, \dots, d-1$ , it follows that

$$\varepsilon[\zeta(t)\tilde{x}^T(t,1)] = \varepsilon[\tilde{x}(t_d,2)\tilde{x}^T(t,1)] = M_1(0,d). \quad (48)$$

Further, note (46), we have

$$\begin{aligned}
& \varepsilon[\zeta(t)\zeta^T(t)] \\
& = P_2 - M_2(0,0)H_2^T Q_w^{-1} H_2 [M_2(0,0)]^T
\end{aligned} \quad (49)$$

$$- \sum_{i=1}^{d-1} M_1(0,i)H_1^T Q_w^{-1}(i,1)H_1 [M_1(0,i)]^T.$$

Substituting (47), (48) and (49) into (46), (41) can be easily obtained.  $\square$

Now we are to calculate  $K_i$  based on the reorganized innovation approach in the following theorem.

**Theorem 4:** The  $K_0$  is computed by

$$K_0 = [M_1(d,d-1) \quad S] \begin{bmatrix} H_{(0)} & 0 \\ 0 & H_{(1)} \end{bmatrix}^T Q_w^{-1}, \quad (50)$$

where

$$\begin{aligned}
S & = M_2(d+1,0) \\
& \quad - \sum_{j=0}^1 M_2(d+j,0)H_2^T Q_w^{-1} H_2 [M_2(j-1,0)]^T \\
& \quad - \sum_{j=1}^{d-2} M_1(d,j)H_1^T Q_w^{-1}(j,1)H_1 [M_1(-1,j)]^T.
\end{aligned}$$

The  $K_i$ ,  $i=1, \dots, d$  is computed by

$$K_i = [M_1(0,d-i) \quad N(i)] \begin{bmatrix} H_{(0)} & 0 \\ 0 & H_{(1)} \end{bmatrix}^T Q_w^{-1}, \quad (51)$$

where

$$\begin{aligned}
N(i) & = M_2(i,0) \\
& \quad - \sum_{j=-i}^0 M_2(-j,0)H_2^T Q_w^{-1} H_2 [M_2(-j-i,0)]^T \\
& \quad - \sum_{j=1}^{d-i-1} M_1(0,j)H_1^T Q_w^{-1}(j,1)H_1 [M_1(-i,j)]^T,
\end{aligned}$$

in the above,  $M_1(\cdot, \cdot)$  and  $M_2(\cdot, \cdot)$  are computed as (40) and (39).  $Q_w(\cdot, 1)$  and  $Q_w$  are calculated via (38) and (36).

**Proof:** Note (45), we have

$$w(t-i) = \begin{bmatrix} H_{(0)} & 0 \\ 0 & H_{(1)} \end{bmatrix} \begin{bmatrix} \tilde{x}(t-i,1) \\ \zeta(t-i) \end{bmatrix} + v_s(t-i), \quad (52)$$

where

$$\begin{aligned}
\tilde{x}(t-i,1) & = x(t-i) - \hat{x}(t-i,1), \\
\zeta(t-i) & = x(t_d-i) - \hat{x}(t_d-i|t-i-1).
\end{aligned}$$

Applying the projection formula and the reorganized innovation, we get

$$\begin{aligned}
 & \zeta(t-i) \\
 &= x(t_d-i) - \text{Pr } \text{oj}\{x(t_d-i) | w(0,2), \dots, w(t_d,2); \\
 & \quad w(t_d+1,1), \dots, w(t-i-1,1)\} \\
 &= x(t_d-i) - \hat{x}(t_d-i,2) \\
 & \quad + \sum_{j=-i}^0 \varepsilon[x(t_d-i)w^T(t_d+j,2)]Q_{w_2}^{-1}w(t_d+j,2) \\
 & \quad - \sum_{j=1}^{d-i-1} \varepsilon[x(t_d-i)w^T(t_d+j,1)] \\
 & \quad \times Q_w^{-1}(j,1)w(t_d+j,1) \\
 &= \tilde{x}(t_d-i,2) \\
 & \quad + \sum_{j=-i}^0 M_2(-j-i,0)H_2^T Q_{w_2}^{-1}w(t_d+j,2) \\
 & \quad - \sum_{j=1}^{d-i-1} M_1(-i,j)H_1^T Q_w^{-1}(j,1)w(t_d+j,1).
 \end{aligned} \tag{53}$$

Noting that  $x(t_d)$  is uncorrelated with  $v_s(t-i)$ ,  $i=0, \dots, d$ . Substituting (52) into (15), it follows that

$$\begin{aligned}
 K_i &= \begin{bmatrix} \varepsilon[x(t_d)\tilde{x}^T(t-i,1)] & \varepsilon[x(t_d)\zeta^T(t-i,1)] \end{bmatrix} \\
 & \quad \times \begin{bmatrix} H_{(0)} & 0 \\ 0 & H_{(1)} \end{bmatrix} Q_w^{-1}.
 \end{aligned} \tag{54}$$

In view of (40), we have that

$$\varepsilon[x(t_d)\tilde{x}^T(t-i,1)] = M_1(0, d-i). \tag{55}$$

Substituting (53) into  $\varepsilon[x(t_d)\zeta^T(t-i,1)]$ , it yields

$$\begin{aligned}
 & \varepsilon[x(t_d)\zeta^T(t-i,1)] \\
 &= \varepsilon[x(t_d)\tilde{x}^T(t_d-i,2)] \\
 & \quad - \sum_{j=-i}^0 \varepsilon[x(t_d)w^T(t_d+j,2)]Q_{w_2}^{-1} \\
 & \quad \times H_2[M_2(-j-i,0)]^T \\
 & \quad - \sum_{j=1}^{d-i-1} \varepsilon[x(t_d)w^T(t_d+j,1)]Q_w^{-1}(j,1) \\
 & \quad \times H_1[M_1(-i,j)]^T \\
 &= M_2(i,0) - \sum_{j=-i}^0 M_2(-j,0)H_2^T Q_{w_2}^{-1} \\
 & \quad \times H_2[M_2(-j-i,0)]^T \\
 & \quad - \sum_{j=1}^{d-i-1} M_1(0,j)H_1^T Q_w^{-1}(j,1) \\
 & \quad \times H_1[M_1(-i,j)]^T.
 \end{aligned} \tag{56}$$

Substituting (56) and (55) into (54), we can prove (51).

By applying the similar approach,  $K_0$  can be easily obtained as (50).  $\square$

**Remark 7:** Theorems 3 and 4 have given the solutions to  $Q_w$  and  $K_i$ ,  $i=0, \dots, d$  based on projection theory in Hilbert space and time-domain reorganized approach. It should be pointed that the reorganized innovation is different from the innovation in Kalman filtering formulation, which is defined in (11).

### 5. NUMERICAL EXAMPLE

In this section, a numerical example will be given to verify the computational procedure of the proposed approach. Consider the systems (1)-(2) with  $d=1$  and

$$\Phi = 0.3, \Gamma = 1, H_{(0)} = 0.5, H_{(1)} = 0.2.$$

Here, we suppose that  $u(t)$ ,  $v_{(0)}(t)$  and  $v_{(1)}(t)$  are mutually uncorrelated white noises with zero means and covariance as 1.

Obviously  $A(q^{-1}) = 1 - 0.3q^{-1}$ ,  $C(q^{-1}) = 1$  and  $B(q^{-1}) = 1$ ,  $D(q^{-1}) = \begin{bmatrix} 0.25 \\ 0.2q^{-1} \end{bmatrix}$ . Applying Theorem 1, we can obtain the spectral factor as follows

$$E(q^{-1}) = \begin{bmatrix} -0.7 + 0.4285q^{-1} & 0.7199 - 0.3847q^{-1} \\ 0.6773 - 0.2869q^{-1} & 0.7545 - 0.1401q^{-1} \end{bmatrix}.$$

In terms of Theorem 2, we solve (26) and obtain  $S(q^{-1}) = [-1.207 \quad 0.7388]$ . By substituting  $E(q^{-1})$  and  $S(q^{-1})$  into (25), thus steady-state optimal filter can be designed. The simulation results are shown in the following Fig. 1.

It can be shown from Fig. 1 that the filter  $\hat{x}(t|t)$

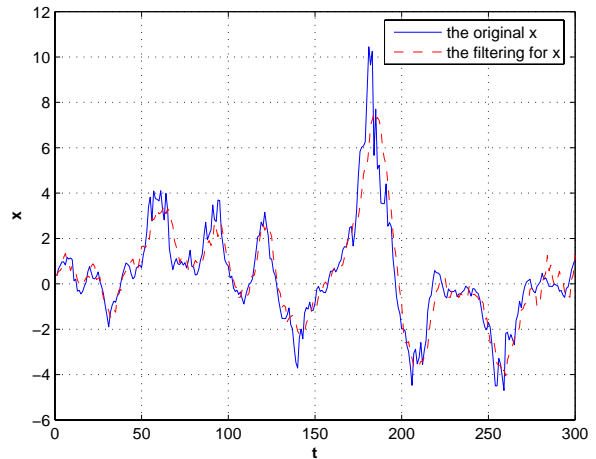


Fig. 1. Tracking performance of the filter  $\hat{x}(t|t)$ .

can track the original state  $x(t)$  very well, which implies the presented method in this paper is effective.

## 6. CONCLUSION

We have studied the steady-state optimal filtering problem for discrete-time systems with instantaneous measurement and single delayed measurement. By applying the time-domain reorganized innovation approach, based on ARMA innovation model, spectral factorization is easily calculated. The calculation does not require the state augmentation. Thus the steady-state optimal filter is designed via one spectral factorization and one Diophantine equation. The key technique that is applied in this paper is the reorganized innovation analysis approach in Hilbert space.

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